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Geometrical Aspects of Skyrmions, Reflection Group, and the Internal Symmetry of Hadrons

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The topological aspects of skyrmions are studied and it is shown that hadrons can be viewed as composite states of baby skyrmions when the internal symmetry $SU(3)$ is generated from reflection. It is shown that in an anisotropic space a particle can move with $l = 1/2$ with a specific l_x value, and a bosonic constituent moving with $l = 1/2$ will appear as a baby skyrmion and a fermionic constituent will appear as if a spin carrier is attached to a baby skyrmion. The associated magnetic field causes a strong statistical attraction which helps to form the bound state of such constituents. The doublet of such particles having opposite l_x values form a conformal spinor when each member behaves as a Cartan semispinor. The conformal reflection then helps us to generate the internal $SU(3)$ symmetry, which splits as $SU(3) \rightarrow SU(2) \times U(1)$, giving rise to the hadronic spectra. The strong interaction involves a composite cluster in such a bound system when rearrangement of the constituents takes place preserving the direction vectors, and an elementary constituent can take part in a weak interaction, causing parity violation. These features help us to consider elementary constituents as known particles like leptons.

1. INTRODUCTION

The idea of the topological origin of the baryon number proposed by Skyrme [1] and Finkelstein and Rubinstein [2] has been revived. These authors put forward the idea that conserved quantum numbers arise as a consequence of the topological properties of hadrons and that particles which carry conserved quantum numbers are built up from classical fields of nontrivial topology. In this picture, baryons appear as solitons, commonly known as skyrmions. In a recent paper [3], it has been shown that the Skyrme term, which is a square term of the commutator and is necessary for the stability of a soliton, may appear as a consequence of the anisotropic feature of the

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internal space-time such that a ‘direction vector’ is attached to each space-time point, and this property of internal space-time helps us to have a consistent quantization of a Fermi field. In this scheme, all massive fermions appear as solitons and the Skyrme term may be considered as an effect of quantization. However, the Skyrme term does not manifestly express the internal anisotropy as it is invariant under *P* and *T*. So to incorporate this anisotropic feature in the Lagrangian, we should add the Wess–Zumino term, which is a five-dimensional integral having a boundary denoting our physical space-time. Witten [4] has shown that the constant appearing in the Wess–Zumino action has to be an integer for the existence of a consistent quantum description, which is analogous to the Dirac quantization of a product of electric and magnetic charges.

The quantization procedure of a fermion suggests that we should take into account an anisotropic feature in the microlocal space-time where a ‘direction vector’ ξ_μ is attached to the space-time point x_μ, and this gives rise to an internal helicity corresponding to the fermion number [3]. The introduction of this ‘direction vector’ can be transcribed in terms of spinorial variables attached to space-time points which give rise to a $SL(2, C)$ gauge-theoretic description of the fermion, which is described by a nonlinear σ-model with a Wess–Zumino term. The $SL(2, C)$ gauge fields give rise to the Pontryagin index, which appears as a magnetic charge and is responsible for the topological origin of fermion number. This gauge field current associates a magnetic field which gives rise to a strong attractive interaction in a composite system analogous to the effect of the Chern–Simons term in a (2 + l)-dimensional system. Indeed, the anisotropic feature of the internal space helps us to modify the angular momentum in the same way as a magnetic monopole changes the angular momentum of a charged particle, and this shift in angular momentum causes a shift in statistics. In fact, in such a coordinate system a particle can move with $l = 1/2$ with a specific l_x value, and a bosonic constituent moving with $l = 1/2$ will appear as a baby skyrmion and a fermionic constituent will appear as if a spin carrier is attached to a baby skyrmion. The associated magnetic field causes a strong statistical attraction which helps to form the bound state of such constituents. The doublet of such particles having opposite l_x values form a conformal spinor when each member behaves as a Cartan semispinor. The two members of the doublet correspond to particle and antiparticle states in Minkowski space representing baby skyrmion and antiskyrmion. The conformal reflection then helps us to generate the internal symmetry $SU(3)$, which splits as $SU(3) \rightarrow SU(2) \times U(1)$, giving rise to the hadronic spectra. Thus we can have a geometrical origin of the internal $SU(3)$ symmetry of hadrons when these are depicted as composite systems where the elementry constituents are bound by a strong statistical attraction caused by the associated magnetic field. This then helps us to consider the elementary constituents of hadrons as known particles like leptons.

In Section 2 we describe the topological aspects of a skyrmion and the geometrical setup for a composite system formed by statistical interaction. In Section 3 we discuss the geometrical origin of $SU(3)$ symmetry from the reflection group. Section 4 sketches a possible configuration scheme of hadrons based on this formalism and Section 5 considers the static properties of baryons. In Section 6 we discuss hadronic interactions and various selection rules.

2. TOPOLOGICAL ASPECTS OF A SKYRMION, COMPOSITE SYSTEM, AND STATISTICAL INTERACTION

In an earlier paper [3] we showed that the quantization of a Fermi field is achieved when an anisotropy in the internal space is introduced so that it gives rise to two helicities of opposite orientations corresponding to fermion and antifermion. To have quantization in Minkowski space, we have to take into account a complex manifold where the coordinate is given by $z^\mu = x^\mu + i\xi^\mu$, where ξ^μ is the four-vector in the internal space [5]. The anisotropic feature of this ξ-space helps us to consider it as an attached 'direction vector' to the space-time point x_μ so that the two opposite orientations of the 'direction vector' give rise to fermion and antifermion. This helps us to have a gauge-theoretic extension of a relativistic quantum particle when the gauge group is given by $SL(2, C)$. This inherent gauge structure seems to be the major ingredient of the quantization procedure.

In the case of a massive spinor we can choose the chiral coordinate as

$$z^\mu = x^\mu + \frac{i}{2}\lambda^\mu_\alpha \theta^\alpha \tag{1}$$

where we identify the coordinate in the complex manifold $z^\mu = x^\mu + i\xi^\mu$ with $\xi^\mu = \frac{1}{2}\lambda^\mu_\alpha\theta^\alpha$ ($\alpha = 1, 2$), θ being a two-component spinor. We now replace the chiral coordinates by the matrices

$$z^{AA'} = x^{AA'} + \frac{i}{2}\lambda^{AA'}_\alpha \theta^\alpha \tag{2}$$

where

$$x^{AA'} = \frac{1}{\sqrt{2}}\begin{bmatrix} x^0 - x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + x^1 \end{bmatrix}$$

and

$$\lambda_\alpha^{AA'} \in SL(2, C)$$

With these relations, the twistor equation is now modified as [6]

$$\bar{Z}_\alpha Z^\alpha + \lambda_\alpha^{AA'}\theta^\alpha \bar{\pi}_A \pi_{A'} = 0 \tag{3}$$

where $\bar{\pi}_A(\pi_{A'})$ is the spinorial variable corresponding to the four-momentum variable p^μ, the conjugate of x^μ, and is given by the matrix representation

$$p^{AA'} = \bar{\pi}^A \pi^{A'} \tag{4}$$

and $Z^\alpha = (\omega^A, \pi_{A'})$, $\bar{Z}_a = (\bar{\pi}_A, \bar{\omega}^{A'})$ with $\omega^A = i(x^{AA'} + \frac{i}{2}\lambda_\alpha^{AA'}\theta^\alpha)\pi_{A'}$. Now Eq. (3) involves the helicity operator

$$S = -\lambda_\alpha^{AA'}\theta^\alpha \bar{\pi}_A \pi_{A'} \tag{5}$$

which we identify as the internal helicity and corresponds to the fermion number. It may be noted that we have taken the matrix representation of p_μ, the conjugate of x_μ in the complex coordinate $z_\mu = x_\mu + i\xi_\mu$, as $p^{AA'} = \bar{\pi}^A \pi^{A'}$, implying $p_\mu^2 = 0$, and so the particle will have mass due to the nonvanishing character of the quantity ξ_μ^2. It is observed that the complex conjugate of the chiral coordinate given by (2) will give rise to a massive particle with internal helicity of opposite orientation corresponding to an antifermion. In the null plane where $\xi_\mu^2 = 0$, we can write the chiral coordinate as

$$z^{AA'} = x^{AA'} + \frac{i}{2}\bar{\theta}^A \theta^{A'} \tag{6}$$

where the coordinate ξ^μ is replaced by $\xi^{AA'} = \frac{1}{2}\bar{\theta}^A\theta^{A'}$. In this case the helicity operator is given by

$$S = -\bar{\theta}^A \theta^{A'} \bar{\pi}_A \pi_{A'} = -\bar{\varepsilon}\varepsilon \tag{7}$$

where $\varepsilon = i\theta^{A'}\pi_{A'}$ and $\bar{\varepsilon} = -i\bar{\theta}^A\bar{\pi}_A$. Shirafuji [7] noted that the state with the helicity $+1/2$ is the vacuum state of the fermion operator

$$\varepsilon|s = +1/2\rangle = 0 \tag{8}$$

Similarly the state with helicity $-1/2$ is the vacuum state of the fermion operator

$$\bar{\varepsilon}|s = -1/2\rangle = 0 \tag{9}$$

From this analysis, it is noted that we can define a plane D^- where for the coordinate $z_\mu = x_\mu + i\xi_\mu$, ξ_μ belongs to the interior of the forward light cone $\xi >> 0$ and as such represents the upper half-plane with the condition det

$\xi > 0$ and $\frac{1}{2}Tr\xi > 0$. The lower half-plane D^{+} is given by the set of all coordinates z_{μ} with ξ_{μ} in the interior of the backward light cone. The map $z \rightarrow z^{*}$ sends the upper half-plane to the lower half-plane. The space M of the null plane ($\det \xi = 0$) is the Shilov boundary, so that a function holomorphic in D^{-} (D^{+}) is determined by its boundary values. Thus if we consider that any function $\phi(z) = \phi(x) + i\phi(\xi)$ is holomorphic in the whole domain, the helicity $+1/2$ ($-1/2$) given by the operator $i\theta^{A'}\ \pi_{A'}$ ($-i\bar{\theta}^{A}\bar{\pi}_{A}$) in the null plane may be taken to be the limiting value of the internal helicity in the upper (lower) half-plane.

In the sense of Minkowski space-time, the characteristics $\xi >> 0$ and $\xi << 0$ in the upper and lower half-planes indicate that the domain is disconnected and anisotropic in nature. This indicates that the behavior of the angular momentum operator in such a region will be similar to that of a charged particle moving in the field of a magnetic monopole. In fact, in such a case, the wave function given by $\phi(z_{\mu}) = \phi(x_{\mu}) + i\phi(\xi_{\mu})$ can be treated as describing a particle moving in the external spaced-time having the coordinate x_{μ} with an attached 'direction vector' ξ_{μ}. Thus the wave function should take into account the polar coordinates r, θ, ϕ along with the angle χ specifying the rotational orientation around the 'direction vector' ξ_{μ}. The eigenvalue of the operator $\partial/\partial\chi$ just corresponds to the internal helicity. For an extended body represented by the de Sitter group $SO(4,1)$, θ, ϕ, and χ just represent the three Euler angles.

In 3-space, these three Euler angles correspond to an axisymmetric system where the anisotropy is introduced along a particular direction and the components of the linear momentum satisfy a commutation relation of the form

$$[p_i, p_j] = i\mu\epsilon_{ijk}\frac{x^k}{r^3} \tag{10}$$

where μ represents the measure of anisotropy and is given by the commutation relation, suggesting that it behaves like the strength of a magnetic monopole. The angular momentum operator $\vec{J}$ is given by

$$\vec{J} = \vec{r} \times \vec{p} - \mu\vec{r} \tag{11}$$

The spherical harmonics incorporating the term μ have been extensively studied by Fierz [8] and Hurst [9]. Following them, we write

$$Y_l^{m,\mu} = (1 + x)^{-(m-\mu)/2}(1 - x)^{-(m+\mu)/2} \times \frac{d^{l-m}}{d^{l-m}x}[(1 + x)^{l-\mu}(1 - x)^{l+\mu}]e^{i(m-\mu)\phi} \tag{12}$$

with $x = \cos\theta$. The quantities m and μ represent the eigenvalues of the operators $i\partial/\partial\phi$ and $i\partial/\partial\chi$, respectively, when the wave function is written in terms of the angles θ, ϕ, and χ. For $m = \pm 1/2$, $\mu = \pm 1/2$, we have

$$\begin{aligned} Y_{1/2}^{1/2,1/2} &= \sin(\theta/2)\, e^{i/2(\phi-\chi)} \\ Y_{1/2}^{-1/2,1/2} &= \cos(\theta/2)\, e^{-i/2(\phi+\chi)} \\ Y_{1/2}^{1/2,-1/2} &= \cos(\theta/2)\, e^{i/2(\phi+\chi)} \\ Y_{1/2}^{-1/2,-1/2} &= \sin(\theta/2)\, e^{-i/2(\phi-\chi)} \end{aligned} \tag{13}$$

These represent spherical harmonics for half-orbital angular momentum $l = 1/2$ with $\mu = \pm 1/2$. An important feature of this formalism is that a particle can move with $l = 1/2$ in the internal space of a composite system with the specific property that $l_x = +1/2$ and $-1/2$ corresponds to the internal helicity or orientation [6]. Note that the motion of a particle in an anisotropic space gives rise to similar features as that of a charged particle in the field of a magnetic monopole. This becomes evident from the angular momentum relation (11), where μ can be viewed as the monopole charge. The fact that in such an anisotropic space the angular momentum can take the value 1/2 is then found to be analogous to the result that a monopole–charged particle composite representing a dyon satisfying the condition $eg = 1/2$ has angular momentum shifted by 1/2 unit and statistics shift accordingly [10]. A fermion (boson) moving with $l = 1/2$ will be transformed into a boson (fermion).

In this complexified space-time exhibiting the internal helicity states, we can write the metric $g_{\mu\nu}(x, \theta, \bar{\theta})$. It has been shown elsewhere [11] that this metric structure gives rise to the $SL(2, C)$ gauge theory of gravitation and generates the field strength tensor $F_{\mu\nu}$ given in terms of the gauge fields B_μ, which are matrix-valued, having the $SL(2, C)$ group structure, and is given by

$$F_{\mu\nu} = -\partial_\nu B_\mu + \partial_\mu B_\nu + [B_\mu, B_\nu] \tag{14}$$

Since θ $(\bar{\theta})$ is the spinorial variable which represents the ‘direction vector’ attached to the space-time point, these effectively represent the extension of a relativistc quantum particle representing a fermion and we can reformulate it as a gauge-theoretic extension of a relativistic particle with matrix-valued non-Abelian gauge fields having the $SL(2, C)$ group structure.

In fact, we can write for the relativistic extension of a quantum particle the position and momentum variables as [5]

$$\begin{aligned} Q_\mu &= -i(\partial/\partial p_\mu + B_\mu) \\ P_\mu &= i(\partial/\partial q_\mu + C_\mu) \\ B_\mu, C_\mu &\in SL(2, C) \end{aligned} \tag{15}$$

Now we note that if we demand $F_{\mu\nu} = 0$ at all points on the boundary S^3 of a certain volume V^4 inside which $F_{\mu\nu} \neq 0$, the gauge potentials tend to a pure gauge in the limit toward the boundary

$$B_\mu = U^{-1}\, \partial_\mu\, U \tag{16}$$

Thus we write in this limiting case

$$L = M^2\, Tr(\partial_\mu U^{-1}\, \partial_\mu U + Tr[\partial_\mu UU^\dagger,\, \partial_\nu UU^\dagger]^2 \tag{17}$$

where M is a suitable constant having the dimension of mass. It is noted that the Skyrme term $Tr[\partial_\mu UU^\dagger, \partial_\nu UU^\dagger]^2$ arises here from the term $F_{\mu\nu}F^{\mu\nu}$, where the first term is related to the gauge-noninvariant term $M^2 B_\mu B^\mu$ in the Lagrangian. Thus we find that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity giving rise to fermion number corresponds to the realization of a nonlinear σ-model, where the Skyrme term introduced for stabilization of the soliton automatically arises here as an effect of quantization. Thus in this picture massive fermions appear as solitons and the fermion number is of topological origin. Indeed, for the Hermitian representation, we can take the group manifold as $SU(2)$ and this leads to a mapping from the space 3-sphere S^3 to the group space S^3 [$SU(2) = S^3$]; the corresponding winding number is given by

$$q = \frac{1}{24\pi^2} \int dS_\mu\, \epsilon^{\mu\nu\alpha\beta}\, Tr[U^{-1}\partial_\nu UU^{-1}\partial_\alpha UU^{-1}\partial_\beta U] \tag{18}$$

It is noted that the Skyrme term which arises here as an effect of quantization does not manifestly express the internal anisotropy as it is invariant under P and T. So to incorporate this anisotropic feature in the Lagrangian, we should add the Wess–Zumino term, where the action is given by

$$S_{\mathrm{WZ}} = \frac{iN}{240\pi^2} \int_B d^5x\, \epsilon^{\mu\nu\alpha\beta\gamma}\, Tr[U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}\partial_\alpha UU^{-1}\partial_\beta UU^{-1}\partial_\gamma U]$$
$$x = \vec{x},\, t,\, x^5 \tag{19}$$

Here the physical space-time is the boundary of the five-dimensional domain.

Noted that if we demand $SL(2, C)$ invariance in spinor affine space, the simplest Lagrangian density which is invariant under $SL(2, C)$ transformation is given by

$$L = \frac{-1}{4}\, Tr\, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \tag{20}$$

which violates P and T and thus finds its correspondence with the Wess–Zumino term in the Skyrme Lagrangian. Following Carmeli and Malin [12], if we apply the usual procedure of variational calculus, we get the field equations

$$\partial_\delta(\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}) - [B_\delta, \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}] = 0 \tag{21}$$

Taking the infinitesimal generators of the group $SL(2, C)$ in tangent space as

$$g_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad g_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{22}$$

we can write

$$\begin{aligned} B_\mu &= B^a_\mu g^a = \vec{B}_\mu \cdot \vec{g} \\ F_{\mu\nu} &= F^a_{\mu\nu} g^a = \vec{F}_{\mu\nu} \cdot \vec{g} \end{aligned} \tag{23}$$

Evidently, in this space, these $SL(2, C)$ gauge fields will appear as background fields.

Thus, to describe a matter field in this geometry, the Lagrangian will be modified by the introduction of this $SL(2, C)$-invariant Lagrangian density. Hence, for a massless Dirac field, we write for the Lagrangian

$$L = -\overline{\psi}\gamma_\mu D_\mu \psi - \frac{1}{4} Tr\, \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \tag{24}$$

where D_μ is the $SL(2, C)$ gauge-covariant derivative defined by

$$D_\mu = \partial_\mu - igB_\mu \tag{25}$$

where g is some coupling strength.

From this we can construct a conserved current corresponding to this Lagrangian density (neglecting the coupling with the gauge field)

$$\begin{aligned} \vec{j}^\mu &= \overline{\psi}\vec{\gamma}^\mu\psi + \epsilon^{\mu\nu\alpha\beta}\vec{B}_\nu \times \vec{F}_{\alpha\beta} \\ &= \vec{j}^\mu_x + \vec{j}^\mu_\theta \end{aligned} \tag{26}$$

Indeed, we find from (21) that

$$\epsilon^{\mu\nu\alpha\beta}[\partial_\nu\vec{F}_{\alpha\beta} - \vec{B}_\nu \times \vec{F}_{\alpha\beta}] = 0 \tag{27}$$

This suggests that

$$\begin{aligned} \vec{j}^\mu_\theta &= \epsilon^{\mu\nu\alpha\beta}\vec{B}_\nu \times \vec{F}_{\alpha\beta} \\ &= \epsilon^{\mu\nu\alpha\beta}\partial_\nu\vec{F}_{\alpha\beta} \end{aligned} \tag{28}$$

This gives

$$\partial_\mu\vec{j}^\mu_\theta = \epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\vec{F}_{\alpha\beta} = 0 \tag{29}$$

However, in the Lagrangian (24), if we split the Dirac massless spinor into chiral forms and identify the internal helicity with left (right) chirality corresponding to θ ($\bar{\theta}$), we have the following conservation laws [13]

$$\partial_\mu\left[\frac{1}{2}\,(-ig\bar{\psi}_R\gamma_\mu\psi_R + j^1_\mu\right] = 0$$

$$\partial_\mu\left[\frac{1}{2}\,(-ig\bar{\psi}_L\gamma_\mu\psi_L + ig\bar{\psi}_R\gamma_\mu\psi_R) + j^2_\mu\right] = 0$$

$$\partial_\mu\left[\frac{1}{2}\,(-ig\bar{\psi}_L\gamma_\mu\psi_L) + j^3_\mu\right] = 0 \tag{30}$$

These three equations represent a consistent set of equations if we choose

$$j^1_\mu = -j^2_\mu/2; \qquad j^3_\mu = +j^2_\mu/2 \tag{31}$$

which evidently guarantees the vector current conservation. Then we can write

$$\partial_\mu(\bar{\psi}_R\gamma_\mu\psi_R + j^2_\mu) = 0$$
$$\partial_\mu(\bar{\psi}_L\gamma_\mu\psi_L - j^2_\mu) = 0 \tag{32}$$

From these, we have

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi) = \partial_\mu j^5_\mu = -2\partial_\mu j^2_\mu \tag{33}$$

Thus the anomaly is expressed here in terms of the second $SL(2, C)$ component of the gauge field current j^2_μ. However, since in this formalism the chiral currents are modified by the introduction of j^2_μ, we note from (32) that the anomaly vanishes.

The charge corresponding to the gauge field part is

$$q = \int j^2_0\, d^3x = \int_{\text{surface}} \epsilon^{ijk} d\sigma_i F^2_{jk} \qquad (i, j, k = 1, 2, 3) \tag{34}$$

Visualizing F^2_{jk} to be the magnetic field-like components for the vector potential B^2_i, we see that q is actually associated with the magnetic pole strength for the corresponding field distribution. Thus we find that the quantization of a Fermi field associates a background magnetic field, and the charge corresponding to the gauge field current effectively represents a magnetic charge.

The term $\epsilon^{\alpha\beta\gamma\rho}F_{\alpha\beta}F_{\gamma\rho}$ in the Lagrangian can be expressed as a four-divergence of the form $\partial_\mu\Omega^\mu$, where

$$\Omega^\mu = -\frac{1}{16\pi^2}\,\epsilon^{\mu\alpha\beta\gamma}\,Tr\left[B_\alpha F_{\beta\gamma} - \frac{2}{3}\,(B_\alpha B_\beta B_\gamma)\right] \tag{35}$$

We recognize that the gauge field Lagrangian is related to Pontryagin density

$$P = -\frac{1}{16\pi^2} Tr{}^*F_{\mu\nu}F_{\mu\nu} = \partial_\mu \Omega^\mu \tag{36}$$

where Ω^μ is the corresponding Chern–Simons secondary characteristic class. The Pontryagin index

$$q = \int P\, d^4x \tag{37}$$

is then a topological invariant. As we know, the introduction of the Chern–Simons characteristic class modifies the axial vector current as

$$\tilde{j}^5_\mu = j^5_\mu + 2\hbar\Omega_\mu \tag{38}$$

where $\partial_\mu \tilde{j}^5_\mu = 0$, though $\partial_\mu j^5_\mu \neq 0$; we find from (33) that the Chern–Simons characteristic class is effectively represented by the current constructed from the $SL(2, C)$ gauge field. Thus we have the Chern–Simons topology built into the system and is associated with the topological aspects of a fermion arising out of the quantization procedure. We find that the origin of the Wess–Zumino term in the Skyrme Lagrangian is associated with the P and T-violating gauge field current j^2_μ. From this analysis, we note that the gauge field current associates a magnetic field with a fermion in $3 + 1$ dimensions. Indeed, we pointed out that the quantization procedure of a fermion suggests that a 'direction vector' is attached to a space-time point and the orientation of this 'direction vector' is associated with the fermion number. This helps us to have a topological origin of the fermion number associated with the Pontryagin index, which effectively represents the magnetic charge arising out of the background $SL(2, C)$ gauge fields. As discussed earlier, this picture of a fermion has its correspondence with a scalar particle moving in an anisotropic space with $l = 1/2$ with a specific l_z value. The inherent anisotropic feature of space will associate a magnetic field with such a particle and the classical free particle current will be modified by the background gauge field current j^2_μ representing the contribution of the associated magnetic field. Evidently this will correspond to a chiral fermion. However, the background magnetic field energy $\int B^2\, d^3x$ will modify the free-particle mass of such a particle. Even if such a particle has its free mass zero, the magnetic field energy will contribute to the mass of such a particle and it will behave as a massive particle. The present analysis suggests that such a particle may be considered as a skyrmion represented by a nonlinear σ-model with Wess–Zumino term.

If we consider a composite system such that the space-time coordinate of each constituent has an attached 'direction vector' (vortex line), we may view it as if it is moving with $l = 1/2$ in an anisotropic space with a specific l_z value. This implies that a bosonic constituent will behave as if it is a skyrmion and a fermion will behave as if a spin carrier is attached to a baby skyrmion. Evidently this will transform a fermionic (bosonic) constituent into a bosonic (fermionic) one. So, for a fermion, we will have the centrifugal barrier minimized, denoting a strong attractive interaction. Thus we note that in a composite system, if we consider that the internal space is anisotropic in nature so that a constituent can move with $l = 1/2$ with a specific l_z value, the associated magnetic field will generate a strong attractive statistical interaction and this will help us to have stable bound states. In the following sections, we shall depict hadrons as the bound states of such systems where the internal $SU(3)$ symmetry is generated from geometrical considerations.

3. BABY SKYRMIONS, REFLECTION GROUP, AND THE INTERNAL SYMMETRY OF HADRONS

It is well known that the wave function of the form $\phi(x_\mu, \xi_\mu)$ where ξ is an attached vector extends the Lorentz group $SO(3, 1)$ to the de Sitter group $SO(4, 1)$. The irreducible representations of $SO(4)$, the maximal compact subgroup of $SO(4, 1)$, are characterized by two numbers (k, n), where k is an integer or half-integer and n is a natural number. These two numbers are related to the values of the Casimir operators

$$\frac{1}{2} S^{\alpha\beta} S_{\alpha\beta} = k^2 + (|k| + n)^2 - 1$$

$$\frac{1}{8} \epsilon^{\alpha\beta\gamma\delta} S_{\alpha\beta} S_{\gamma\delta} = k(|k| + n) \tag{39}$$

where $S_{\alpha\beta}$, α, $\beta = 1, 2, 3, 4$, are the generators of the group. Barut and Bohm [14] have shown that the representation of $SO(4, 1)$ given by $s = 1/2$, $k = \pm 1/2$ can be fully extended to two inequivalent representations of the conformal group $SO(4, 2)$. In fact, these k values actually correspond to the eigenvalues of the operator $k = \frac{1}{2}(a^\dagger a - b^\dagger b)$ in the oscillator representation of the $SO(3)_1 \times SO(3)_2$ basis of $SO(4)$. The value of k as well as its signature is an $SO(4, 2)$ invariant. The representation $s = 0$, $k = 0$ in the conformal representation of $SO(4, 2)$ describes the massless spin-0 particle. The representation $s = 1/2$, $k = \pm 1/2$ describes the helicity state of a massless spinor. Now for a massive spinor, the conformal invariance breaks down and the values $k = \pm 1/2$ then represent internal helicity states so that the two opposite orientations correspond to particle and antiparticle. In the complex manifold with the coordinate $z_\mu = x_\mu + i\xi_\mu$, if we take the wave function $\phi(z_\mu) = \phi(x_\mu) + i\phi(\xi_\mu)$, the inherent disconnected nature of the attached vector ξ_μ for a massive spinor allows us to write

$$\phi(\xi_\mu) = \phi_+(\xi_\mu) + \phi_-(\xi_\mu) \tag{40}$$

where $\phi_+(\xi_\mu)$ $[\phi_-(\xi_\mu)]$ is defined in the upper (lower) half-plane characterised by the fact that ξ_μ belongs to the interior of the forward (backward) light cone with the space M specified by $\|\xi_\mu\|^2 = 0$ representing the boundary. Evidently these two domains are characterized by the internal helicity $k = +1/2$ $(-1/2)$ representing the particle (antiparticle) state. Again, as this fermionic feature is realized when a scalar particle moves in an anisotropic space with $l = 1/2$ having a specified l_z value, we note that the internal helicities given by the k values $+1/2$ and $-1/2$ effectively represent the two l_z values for such a system, which can be described as a baby skyrmion (antiskyrmion).

Since these representations can be fully extended to the conformal group $SO(4, 2)$, we can deal with the eight-component conformal spinors. The simplest conformally covariant spinor field equation postulated as an $SO(4, 2)$-covariant equation in a pseudo-Euclidean manifold $R^{4,2}$ is of the form

$$\left(\Gamma_a \frac{\partial}{\partial \eta_a} + m\right)\xi(\eta) = 0, \qquad a = 0, 1, 2, 3, 5, 6 \tag{41}$$

When the elements of the Clifford algebra Γ_a are the basis unit vectors of $R^{4,2}$, m is a constant matrix and $\xi(\eta)$ is the eight-component spinor field. Cartan [15] has shown that in the fundamental representation where the unit vectors are represented by 8×8 matrices of the form

$$\Gamma_a = \begin{vmatrix} 0 & \Xi \\ H & 0 \end{vmatrix} \tag{42}$$

the conformal spinors ξ are of the form

$$\xi = \begin{vmatrix} \phi_1 \\ \phi_2 \end{vmatrix} \tag{43}$$

where ϕ_1 and ϕ_2 are Cartan semispinors. The characteristic property of these spinors is that for any reflection, ϕ_1 and ϕ_2 interchange. In this basis, Eq. (41) becomes equivalent in Minkowski space $R^{3,1}$ to the coupled equations

$$\begin{aligned} i\,\partial\!\!\!/\phi_1 &= -m\phi_2 \\ i\,\partial\!\!\!/\phi_2 &= -m\phi_1 \end{aligned} \tag{44}$$

However, it is also possible to obtain from Eq. (41), a pair of standard Dirac equations in Minkowski space. To this end, we have to act with a unitary transformation C_1 given by

$$C_1 = \begin{vmatrix} L & R \\ R & L \end{vmatrix} \tag{45}$$

where $L = \frac{1}{2}(1 + \gamma_5)$, $R = \frac{1}{2}(1 - \gamma_5)$ with $\gamma_5 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$. Using this, we have

$$C_1\xi = \xi^D = \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix} \tag{46}$$

and

$$C_1^{-1}\Gamma_\mu C_1 = \Gamma_\mu^D = \begin{vmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{vmatrix} \tag{47}$$

This suggests that Eq. (41) is equivalent in Minkowski space to the pair of standard Dirac equations

$$\begin{aligned} (i\,\tilde{\partial\!\!\!/} + m)\psi_1 &= 0 \\ (i\,\tilde{\partial\!\!\!/} + m)\psi_2 &= 0 \end{aligned} \tag{48}$$

It is to be noted that the space or time reflection interchanges ϕ_1 and ϕ_2 and transforms ψ_1 and ψ_2 into themselves. Conformal reflection interchanges both $\phi_1 \leftrightarrow \phi_2$ and $\psi_1 \leftrightarrow \psi_2$. It should be added that ψ_1 and ψ_2 may represent physical free massive fermions, whereas ϕ_1 and ϕ_2 do not unless they are massless since they obey the coupled equations. However, in the case of $m \neq 0$ if we define ϕ_1 and ϕ_2 such that they represent two different 'internal helicity' states given by $k = +1/2$ and $-1/2$, i.e., $\phi_1 = \psi(k = +1/2)$ and $\phi_2 = \psi(k = -1/2)$, Eqs. (44) can be reduced to a single equation with two internal degrees of freedom when the linear combination of $\psi(k = +1/2$ and $\psi(k = -1/2)$ represents an eigenstate. Now to retain the four-component nature of the spinor in Minkowski space, these two internal degrees of freedom should be associated with particle–antiparticle states. Evidently, this property of ϕ_1 and ϕ_2 satisfies the criteria that the space, time, or conformal reflection changes into one another. This follows from the fact that the parity operator changes the sign of k. Besides, the time reversal changes the orientation of the the internal helicity and hence changes the sign of k. Moreover, the conformed reflection changes one into the other. Thus each member of the doublet of massive spinors having the internal helicity $k = +1/2$ and $-1/2$ and corresponding to the particle and antiparticle states represents a Cartan semispinor. Evidently for the case of a baby skyrmion as this is represented as a scalar particle moving in an anisotropic space with $l = 1/2$ having a specific l_z value, this l_z value ($+1/2$ or $-1/2$) effectively represents the internal helicity k ($+1/2$ or $-1/2$). So if we consider a doublet of baby skyrmion and antiskyrmion having $l_z = +1/2$ and $-1/2$, P, T, as well as conformal reflection will change such a skyrmion into an antiskyrmion and each member will represent a Cartan semispinor.

To have a geometrical interpretation of these spinors, one may look at the totally isotropic 3-planes of a properly complexified pseudo-Euclidean space $R^{4,2}$. There exist two different families of totally isotropic 3-planes which are transformed into one another by a reversal and each is transformed into itself by rotation. The $R^{4,2}$ spinors are isotropic 3-vectors associated with these planes. These can be split into its semispinors $Q = \{{\phi \atop \psi}\}$, where ϕ and ψ are four-component spinors belonging to these two different families. Furthermore, the analysis of Cartan shows that it is possible to regard the components of ϕ as the homogeneous coordinates of a point in 3-dimensional projective space P^3, whereas those of ψ may be regarded as the homogeneous coordinates of a plane in P^3. Moreover, a point–plane correspondence exists in P^3 which reflects the conjugation relation of the semispinors. On the other hand, according to the analysis of Penrose [16], there also exists a 1–1 correspondence between twistors of valence $({1 \atop 0})$ and $({0 \atop 1})$ and point $\leftrightarrow$ plane in P^3. Thus the semispinors into which an eight-component spinor splits in the Cartan basis are identical to the Penrose twistors. This reflects the analysis of Sternberg [17] that charge conjugation corresponds to the Hodge star operation in twistor space.

Now we note that when a fermion moves in the internal space of a system with $l = 1/2$ having a specific l_z value, this can be viewed as if a spin carrier is attached to a baby skyrmion. If we have the constraint that a fermion (antifermion) can only be associated with a baby skyrmion (antiskyrmion), which means that in this system, a fermion can move only with $l_z = +1/2$ (or $-1/2$) and an antifermion can move only with $l_z = -1/2$ (or $+1/2$), then such a particle can be considered as a Cartan semispinor and the doublet will represent a conformal spinor. The induced change in angular momentum and hence in statistics will produce a strong statistical attraction caused by the associated magnetic field and will enable us to form a bound state.

Budinich [18] argued that we can generate an internal symmetry algebra from the conformal reflection group. Budinich suggested that we can call a reflection algebra corresponding to a reflection group an *internal symmetry algebra* for a given field theory if the following hold:

(a) The corresponding reflection group, when accompanied by the corresponding coordinate reflections, is a covariance group for the equation of motion in Minkowski space.

(b) It commutes with the Poincaré Lie algebra and with the space-time reflection algebra.

(c) The transformation induced by the reflection algebra on the fields leaves the action of the theory invariant.

If the reflection algebra commutes only with the Poincaré algebra, but does not commute with the space-time reflection algebra L_4, the algebra may be termed a *restricted internal symmetry algebra*.

To study the conformal reflection algebra, note that since $O(3, 1)$ is a subgroup of $O(4, 2)$, the conformal reflection group will contain as a subgroup the Lorentz reflection group L_4 of four elements

$$L_4 = E, S, T, ST = J \tag{49}$$

where E = identity, S = space reflection, T = time reflection, and $ST = J$ = strong reflection. In $R^{4,2}$ space, coordinates are taken to be η_1, η_2, η_3, η_5, η_0, η_6 with the metric $(+ + + + - -)$; the reflections

$$\begin{aligned} S_5&: \quad \eta_5 \to \eta_5' = -\eta_5 \\ T_6&: \quad \eta^6 \to \eta_6' = -\eta_6 \end{aligned} \tag{50}$$

correspond in Minkowski space to the inverse radius transformation and the same $\otimes J$. Using them, we can build up the four-element Abelian group

$$C_{p6} = E, S_5, T_6, S_5T_6 \tag{51}$$

which is called the partial conformal reflection group. Then the total conformal reflection group, indicated by C_6, is given by the direct product

$$C_6 = C_{p6} \otimes L_4 \tag{52}$$

The conformal reflection group is represented in conformal spinor space by the algebra $U_{4,C}$, which may be called the conformal reflection algebra.

Let ξ be a conformal spinor in the Dirac basis

$$\xi^D = \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix}$$

We know that the Lorentz reflection group L_4 when acting on the Dirac spinor ψ_1 is isomorphic to a U_2 algebra whose Hermitian elements are given by the matrices 1, $i\gamma_0$, $\gamma_0\gamma_5$, γ_5. The transformations S_5, T_6, S_5T_6 when acting on the Dirac doublet of the conformal spinor ξ^D correspond to

$$\begin{aligned} S_5 &\to \Gamma_5^D \\ T_6 &\to i\Gamma_6^D \\ S_5T_6 &\to \Gamma_5^D\Gamma_6^D \end{aligned} \tag{53}$$

Thus the group given by C_{P_o} [Eq. (51)] will be represented by the Lie algebra $U_{2,c}$ and the corresponding real subalgebra $SU(2)$ may be obtained taking

Hermitian elements Γ_5, $i\Gamma_6$, $\Gamma_5\Gamma_6$. Thus the group C_6 is isomorphic to the product

$$U_{2,C} \otimes U_{2,C} = U_{4,C} \tag{54}$$

Then with elegant arguments Budinich proved the following propositions:

(1) The reflection algebra $U_{2,C}$ corresponding to the partial conformal reflection C_{P_6} is an internal symmetry algebra for the conformal spinor doublets. For massive (but degenerate) components of the doublet, $U_{2,C}$ is maximal.

(2) For massless conformal spinors or for a system of massive conformal spinors interacting at very short distances, the direct product of the partial conformal reflection group times the strong reflection in Minkowski space generates a restricted internal symmetry algebra of order eight which can be put in the form $U_{2C,L} \otimes U_{2C,R}$. This algebra may be reduced to two independent $SU(2)$ algebras represented by the eight four-dimensional matrices $L \times \sigma_\mu$, $R \times \sigma_\omega$ $[L = \frac{1}{2}(1 + \gamma_5)$, $R = \frac{1}{2}(1 - \gamma_5)]$ acting on the two independent doublets of Weyl fields into which the massless conformal spinor or the system of interacting massive spinors at short distances splits.

It is to be noted that since reflection is a discrete transformation, we get internal symmetry algebra, but not a group. This difficulty may be avoided if we take the conformal spinor representing a doublet of baby skyrmion and antiskyrmion which moves with $l = 1/2$ having $l_z = +1/2$ and $-1/2$ and characterized by the wave function $\phi(z_\mu) = \phi(x_\mu) + i\phi(\xi_\mu)$, $\phi(\xi_\mu)$ being defined in the domains D^- and D^+, where ξ_μ belongs to the interior of the forward and backward light cones and the space of null plane $\xi_\mu^2 = 0$ is the boundary. Indeed, if we take $\phi(z_\mu)$ as holomorphic in the domains D^- and D^+, the two states with $l_z = +1/2$ and $-1/2$ can be linked through rotation when the angular momentum is given by relation (11). From the above analysis, this will then represent two independent group structures $SU(2)_L \otimes SU(2)_R$. Moreover, the fixed l_z value suggests the existence of the Abelian group $U(1)$. This $SU(2) \otimes U(1)$ group then denotes isospin and hypercharge.

In the harmonic oscillator representation, we can define boson operators for cylindrical coordinates

$$\begin{aligned} a_\pm &= (a_x \mp i a_y)/\sqrt{2} \\ a_0 &= a_z \\ a_\pm^\dagger &= (a_x^\dagger \pm i a_y^\dagger)/\sqrt{2} \\ a_0^\dagger &= a_z^\dagger \end{aligned} \tag{55}$$

In terms of these operators, we can write

$$H = \hbar\omega\{a_+^\dagger a_+ + a_-^\dagger a_- + a_0^\dagger a_0 + 3/2\}$$
$$\lambda_+ = a_+^\dagger a_-$$
$$\lambda_- = a_-^\dagger a_+$$
$$\lambda_0 = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \tag{56}$$

Here the λ-operators are the operators of the two-dimensional oscillator group $SU(2)$, and the two independent $SU(2)$ internal symmetry algebras generated by reflection appear here as the representations of the algebra of this group, which gives rise to isospin. The total isospin operator is given by

$$\lambda^2 = \frac{1}{2}\{\lambda_+\lambda_- + \lambda_-\lambda_+\} + \lambda_0^2 \tag{57}$$

In addition to these isospin operators, we can define the remaining operators of the algebra

$$B_+ = a_+^\dagger a_0, \qquad B_- = a_-^\dagger a_0$$
$$C_+ = a_0^\dagger a_-, \qquad C_- = a_0^\dagger a_+$$
$$N = \frac{1}{3}(a_+^\dagger a_+ + a_-^\dagger a_- - 2a_0^\dagger a_0)$$
$$= \frac{1}{3}(a_x^\dagger a_x + a_y^\dagger a_y - 2a_x^\dagger a_z) \tag{58}$$

The quantum number N is one third the difference between the number of quanta in the X–Y plane and twice the number of quanta in the Z direction. In fact, the operator N corresponds to the hypercharge of the hadron concerned and measures a deformation or departure from spherical symmetry.

The complete classification according to $SU(3)$ and its subgroups $SU(2)$ and $U(1)$ has been given by Elliott [19]. Within a representation of $SU(3)$, the one number representation ϵ of $U(1)$ can take the values

$$\epsilon = 2\lambda + \mu, 2\lambda + \mu - 3, \ldots, -\lambda - 2\mu \tag{59}$$

For a definite representation (λ, μ) and ϵ of $SU(3)$ and $U(1)$, the group $SU(2)$ has representations described by

$$\Lambda = \tfrac{1}{6}|2\lambda - 2\mu - \epsilon|, \tfrac{1}{6}|2\lambda - 2\mu - \epsilon| + 1,$$
$$\ldots \min\{\tfrac{1}{6}(2\lambda + 4\mu - \epsilon), \tfrac{1}{6}(2\lambda + 4\mu + \epsilon)\} \tag{60}$$

The operator $A_{\alpha\alpha}$ $(\alpha = x, y, z) = a_\alpha^\dagger a_\alpha$ simply counts the number of quanta in the α direction. Thus states having a definite number of quanta in each of the three directions in space will have definite value of ν and ϵ, where

$$\nu = N_x - N_y$$
$$\epsilon = 2N_z - N_x - N_y \tag{61}$$

The ϵ and ν values of the many-body system are simply the sum of the ϵ_i and ν_i values of the single-particle constituents

$$\epsilon = \Sigma\, \epsilon_i, \qquad \nu = \Sigma\, \nu_i \tag{62}$$

Now, to find the various ϵ and ν values for the many-particle systems according to the classification of $SU(3) \rightarrow SU(2) \otimes U(1)$, we first form the N-particle function with the maximum possible value of ϵ, $\tilde{\epsilon}^{\max}$, by putting as many particles as allowed by the configuration scheme. It is clear that $\tilde{\epsilon}^{\max} = 2\tilde{\lambda} + \tilde{\mu}$ for the particular representation of $SU(3)$. If the structure of the state having $\epsilon = \tilde{\epsilon}^{\max}$ and $\nu = \tilde{\nu}^{\max}$ is known, other states of the (λ, μ) representation can be constructed using the lowering operators of $SU(3)$. In fact, other ϵ and ν values will be given by relations (59) and (60). In this way, all states classified according to $SU(3) \rightarrow SU(2) \otimes U(1)$ can be constructed. In fact, by chosing ϵ as hypercharge and $\nu/2$ as the third component of isospin, we can find the $SU(3)$ representations of hadronic states.

4. BABY SKYRMIONS, COMPOSITE STATE, AND THE STRUCTURE OF HADRONS

In view of the geometrical origin of the internal $SU(3)$ symmetry from the reflection group and the strong statistical attraction caused by the associated magnetic field, we can consider composite states of baby skyrmions for hadrons, and the spin carriers can be taken to be known particles like leptons. Indeed for this purpose we choose muonic leptons; the motivation behind this will be made clear later.

Let us first consider the configuration $(\mu\nu_\mu)$, where μ represents any of the charge states μ^+, ν_μ, μ^-. We take that μ and ν_μ have an attached direction vector which may be viewed as if it is moving with $l = 1/2$ having a specific l_z value, and the coupling is caused by the strong statistical attraction due to the associated magnetic field. Then by combining the spin and orbital angular momenta of $\mu(\nu_\mu)$, we get J^μ $(J^{(\nu_\mu)}) = 1$ or 0. The total angular momentum J of the system is given by $J = J^\mu + J^{\nu_\mu} + L$, where L is the relative angular momentum, which can take only integer values. This total angular momentum J is nothing but the spin of the particle represented by the composite system. From this we see that the composite system $(\mu\nu_\mu)$ can represent certain types of mesons with spin 0, 1, 2, and so on.

To find the characteristics of these mesons explicitly, let us consider that μ^+ and μ^- stands to each other as particle and antiparticle. Now taking ν_μ as a two-component Weyl spinor (though it will attain mass due to the background magnetic field) and noting that for such a particle 'spin-down' and 'spin-up' states represent the 'particle' and 'antiparticle' states, respectively, if we put the restriction that the $(\mu\nu_\mu)$ system representing a mesonic configuration should have the fermion number zero, μ^+ (μ^-) can then be bound to an antiparticle (particle), i.e., spin-up (spin-down) state only. Furthermore, contending that the configurations $(\mu^+\nu_\mu)$ and $(\mu^-\nu_\mu)$ here stand to each other as particle and antiparticle (or vice versa), we take that the l_z value of ν_μ in these two cases should be of opposite sign and we specify this value in the former and latter cases as $+1/2$ and $-1/2$, respectively. Note that as we have taken that a constituent of a hadron should behave as a Cartan semispinor, if we specify for ν_μ the values $l_z = +1/2$ and $-1/2$ for the states $(\mu^+\nu_\mu)$ and $(\mu^-\nu_\mu)$, respectively, the other states $(\mu^+\nu_\mu)$ with $l_z = -1/2$ and $(\mu^-\nu_\mu)$ with $l_z = +1/2$ are excluded.

Having considered this, we now note that in the configuraton $(\mu\nu_\mu)$, the different values of $J_z^{\nu_\mu}$ are related with the different charge states of the particle μ, so that the charge states of the composite system are completely determined by these values. Then, in the case of $(\mu^+\nu_\mu)$, we have $J_z^{\nu_\mu} = l_z^{\nu_\mu} + s_z^{\nu_\mu} = +1/2 + 1/2 = 1$. Similarly, the case of $(\nu_\mu\nu_\mu)$ and $(\mu^-\nu_\mu)$ we have $J_z^{\nu_\mu} = 0$ and $J_z^{\nu_\mu} = -1$, respectively. So these states $(\mu^+\nu_\mu)$, $(\nu_\mu\nu_\mu)$, and $(\mu^-\nu_\mu)$ can be characterized such that these form a triplet.

Now we show that the doublet of ν_μ's in the systems $(\mu^+\nu_\mu)$ and $(\mu^-\nu_\mu)$ in reality behaves as a conformal spinor and $J_z^{\nu_\mu} = +1$ and -1 for the configurations $(\mu^+\nu_\mu)$ and $(\mu^-\nu_\mu)$ are related by conformal reflection, so that this will represent a 'restricted internal symmetry algebra' $U_{2C,L} \oplus U_{2C,R}$, which will commute with Poincaré algebra, but not with space-time reflection algebra L_4. Indeed, when we consider the doublet

$$\begin{pmatrix} \mu^+ & \nu_\mu \\ \mu^- & \nu_\mu \end{pmatrix}$$

with the constraints $J_z^{\nu_\mu} = +1$ $(l_z^{\nu_\mu} = +1/2,\ s_z^{\nu_\mu} = +1/2)$ for $(\mu^+\nu_\mu)$ and $J_z^{\nu_\mu} = -1 (l_z^{\nu_\mu} = -1/2,\ s_z^{\nu_\mu} = -1/2)$ for $(\mu^-\nu_\mu)$, we note that the doublet $\begin{pmatrix} \nu_\mu \\ \nu_\mu \end{pmatrix}$ represents a conformal spinor $\xi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, each ν_μ acting like a Cartan semispinor having the constraints that the space, time, or conformal reflection transforms $\phi_1 \leftrightarrow \phi_2$. Again, for the neutral configuration $(\nu_\mu\nu_\mu)$ we note that in this system one ν_μ will have $l_z = +1/2$ and $s_z = -1/2$ and the other will have the opposite values, and in this case particle–antiparticle states will be indistinguishable. However, for such a neutral configuration we may take that the constituent fermions are oppositely charged particles such as $(\mu^-\mu^+)$. In fact, in the very short distance region (smaller than the Compton wavelength) the internal symmetry algebra $U_{2,L} \otimes U_{2,R}$ can be realized, as we may have in the Lagrangian density bilinear spinor densities of the type $\bar{\psi}_L\gamma_\mu\psi_L$, $\bar{\psi}_R\gamma_\mu\psi_R$, so that in this very short distance region they split like massless particles. So, as in the massless case, this symmetry $U_{2,L} \oplus U_{2,R}$ gives rise to two independent SU_2 algebras. This suggests that as in the $(\nu_\mu\nu_\mu)$ configuration, in this case also J^{μ^-} or J^{μ^+} may behave as internal symmetry.

From this analysis we note that the J value for ν_μ in the configurations $(\mu^+\nu_\mu)$, $(\nu_\mu\nu_\mu)$, $(\mu^-\nu_\mu)$ given by $J^{\nu_\mu} = l^{\nu_\mu} + s^{\nu_\mu} = 1/2 + 1/2 = 1$ may be taken to represent isospin, where $J_z^{\nu_\mu} = +1$, 0, and -1 is associated with the charge state of the other fermion. For the neutral meson, we may have a configuration like $(\mu^-\mu^+)$ when J^{μ^+} (or J^{μ^-}) represents this isospin. Considering these aspects, we can now identify the three states of the triplet with π-mesons π^+, π^0, and π^-. Also we note that we can have a singlet state having $J^{\nu_\mu} = l^{\nu_\mu} + s^{\nu_\mu} = 1/2 + 1/2 = 0$ corresponding to the neutral state, which is identified with η^0.

It may be noted that a triplet and a singlet of vector (tensor) mesons can also be represented by this configuration scheme with relative angular momentum $L = 1$ (2). We identify these with the triplet of $\rho(A_2)$ mesons and the singlet $\omega^0(f^0)$. It may be added that the mass spectra of spin-0, 1, and 2 neutral isoscalar and isovector mesons are found to be in excellent agreement with experiments when, in the relativistic formulation of the harmonic oscillator framework incorporating the anisotropic nature of the internal space, couplings like $L \cdot (J_1 + J_2)$ and $J_1 \cdot J_2$ are introduced, where $J_i = l_i + s_i$ is the total angular momentum of each constituent [20]. The mass spectrum of charged mesons can then be evaluated by incorporating electromagnetic self-energy. It may be added here that, as is well known, the $\pi^\pm - \pi^0$ mass difference is exactly obtained by this electromagnetic self-energy term.

It is noted from our analysis in the previous section that the geometrical origin of the internal $SU(3)$ symmetry suggests that hadrons having strangeness $+1, 0, -1, -2, -3$ will have different numbers of constituent skyrmions, as $-\epsilon$ represents the hypercharge, and for N-particle states we have $\epsilon = \Sigma\epsilon_i$. A mesonic configuration with strangeness will arise when a neutral baby skyrmion represented by a scalar or pseudoscalar particle moving with $l = 1/2$ having a fixed l_z value is bound to another baby skyrmion represented by a meson which may have any charge state. Identifying the latter constituent as $\pi^+(\pi^0, \pi^-)$ (which are formed by the leptonic constituents as discussed above) and the neutral baby skyrmion as π^0, we note that the configuration of a K-meson may be depicted as $K^+ = (\pi^+\pi^0)$, $K^0 = (\pi^0\pi^0)$ with π^0 moving with $l = 1/2$, $l_z = -1/2$ such that the fixed l_z value of this π^0-meson is associated with the strangeness quantum number. In fact we have $(-1/2)S = l_z^{\pi^0}$, so that $S = +1$. The antiparticle state may be represented by $\bar{K}^0 = (\pi^0\pi^0)$ and $K^- = (\pi^-\pi^0)$ with $l_z^{\pi^0} = +1/2$, so that $S = -1$. To compute isospin, we note that for a composite state of baby skyrmions, this will be given by $J = J^{\pi^+(\pi^0,\pi^-)} + \Sigma J_i$, where $J^{\pi^+(\pi^0,\pi^-)}$ represents the isospin of $\pi^{+(0,-)}$-meson in the configuration given by the J value of the neutral fermion (antifermion) in the pionic configuration as discussed above and J_i is the total angular momentum of other neutral baby skyrmions. Thus the isospin of the K-meson is given by $J = J^{\pi^{+(0,-)}} + J^{\pi^0}$, where $J^{\pi^0} = l^{\pi^0} = 1/2$ is the angular momentum of the neutral π^0 having a fixed l_z value. Indeed we have $J = 1 + 1/2 = 1/2$, so that the fixed l_z value $-1/2$ $(+1/2)$ for particle (antiparticle) state suggests the following J_z values:

$$K^+ = (\pi^+\pi^0) \to J_z = J_z^{\pi^+} + J_z^{\pi^0} = +1 - 1/2 = +1/2$$

$$K^0 = (\pi^0\pi^0) \to J_z = J_z^{\pi^0} + J_z^{\pi^0} = 0 - 1/2 = -1/2$$

Similarly, we will have the antiparticle state when $J_z^{\pi^0} = l_z^{\pi^0} = +1/2$. It may be mentioned that the other possible J value $J = 1 + 1/2 = 3/2$ is forbidden here as the specific l_z value of π^0 does not allow all the possible J_z states.

Note that just like pseudoscalar K-mesons, we can also construct vector (tensor) meson doublets (K^{*+}, K^{*0}), (K^{**+}, K^{**0}) with their antiparticles with the same configuration schemes as the K-meson, with the constraint that the relative angular momentum of the constituents in π^+ (π^0, π^-) is given by $L = 1$ (2) leading to the spin-1 (2) state.

In the case of a baryon we take that a neutral spinor moving with $l = 1/2$, $l_z = +1/2$ is bound to this two-pion configuration of a K-meson. Indeed, denoting this neutral spinor as ν_μ, we note that for a nucleon, configurations like $(\pi^+\pi^0\nu_\mu)$, $(\pi^0\pi^0\nu_\mu)$ suggest that it will have strangeness zero, as $(-1/2)S = l_z^{\pi^0} + l_z^{\nu_\mu} = -1/2 + 1/2 = 0$. Moreover, it will have isospin given by $J = J^{\pi^{+(0)}} + J^{\pi^0} + J^{\nu_\mu}$. Taking $J^{\nu_\mu} = l^{\nu_\mu} + s^{\nu_\mu} = 0$ (1) for spin-1/2 (3/2) baryons, we find for a nucleon $J = 1 + 1/2 + 0 = 1/2$, so that for J_z values we have $J_z = J_z^{\pi^+} (J_z^{\pi^0}) + J_z^{\pi^0} + J_z^{\nu_\mu} = 1\ (0)\ -1/2 + 0 = +1/2\ (-1/2)$ depicting p (n) states. Note that baryon number is associated with the internal helicity given by the fixed l_z value of the spinorial constituent. For an antibaryonic configuration all the constituents will have opposite l_z values.

For the Σ (Λ) hyperon, we consider the configuration $(\pi\pi^0\ \nu_\mu\ \pi^0)$, where another baby skyrmion represented by π^0 moving with $l = 1/2$, $l_z = +1/2$ is bound to the configuration of a nucleon. This will have strangeness $(-1/2)S = l_z^{\pi^0} + l_z^{\nu_\mu} + l_z^{\pi^0} = -1/2 + 1/2 + 1/2 = +1/2$, implying $S = -1$. Isospin is given by $J = J^{\pi} + J^{\pi^0} + J\nu_\mu + J^{\pi^0} = 1 + 1/2 + 0 + 1/2 = 1\ (0)$. The charge states are given by the J_z values with $J_z =$

$J_z^{\pi^+}$ ($J_s^{\pi^0}$, $J_z^{\pi^-}$) + $J_z^{\nu_\mu}$ + $J_z^{\pi^0}$ = 1 (0, −1) − 1/2 + 0 + 1/2 = +1 (0, −1) representing Σ^+ (Σ^0, Σ^-) and for the isosinglet $J = 0$ we will have the neutral configuration depicting Λ. Similarly for the Ξ-particle, we consider the configuration ($\pi\pi^0$ ν_μ π^0 π^0), so that strangeness is given by $(-1/2)S = l_z^{\pi^0} + l_z^{\nu^\mu} + l_z^{\pi^0} + l_z^{\pi^0} = -1/2 + 1/2 + 1/2 + 1/2 = 1$ implying $S = -2$. Isospin is given by $J = J^{\pi} + J^{\pi^0} + J^{\nu_\mu} + J^{\pi^0} + J^{\pi^0} = 1 + 1/2 + 0 + 1/2 + 1/2 = 1/2$ with charge states related to the J_z values where $J_z = J_z^{\pi^0}$ ($J_z^{\pi^-}$) + $J_z^{\pi^0}$ + $J_z^{\nu_\mu}$ + $J_z^{\pi^0}$ + $J_z^{\pi^0}$ = 0 (−1) − 1/2 +0+ 1/2 + 1/2 = +1/2(−1/2) corresponds to the state Ξ^0 (Ξ^-). Note that we cannot have a positive charge state here. It may be added that no other baby skyrmion represented by a π^0-meson moving with $l = 1/2$, $l_z = +1/2$ can be added to this configuration, as this will not give consistent J_z values. This limits the number of spin-1/2 baryons to eight describing the octet representation of $SU(3)$.

This scheme also leads to the decuplet of spin-3/2 baryons. Indeed, taking for the spinor added to the two-pion configuration state the total angular momentum $J = 1$, where for $J_z = +1$, 0, and −1 we have the constraint $l_z = 1/2$ (−1/2) for particle (antiparticle) configuration, we can take the charge states μ^+, ν_μ, and μ^-, respectively. This suggests that $J_z = +1$, 0 (−1, 0) correspond to a baryonic (antibaryonic) configuration and leads to the following charge states having isospin 3/2: $N^{*++} = (\pi^+ \pi^0 \mu^+)$, $N^{*+} = (\pi^+ \pi^0 \nu_\mu)$, $N^{*0} = (\pi^0\pi^0\nu_\mu)$, $N^{*-} = (\pi^-\pi^0\nu_\mu)$. Adding more neutral pions moving with $l = 1/2$ having $l_z = +1/2$, we can construct more baryonic states. Indeed, this will lead to $I = 1$, $S = -1$ states $Y^* = (\pi^+ (\pi^0, \pi^-)\pi^0\nu_\mu\pi^0)$, $I = 1/2$, $S = -2$ state $\Xi^{*0(-)} = (\pi^0(\pi^-)\pi^0\nu_\mu\pi^0\pi^0)$, and the $I = 0$, $S = -3$ state $\Omega^- = (\pi^- \pi^0\nu_\mu\pi^0\pi^0\pi^0)$. Note that no other neutral pion moving with $l = 1/2$ having $l_z = +1/2$ can be added to this configuration, as this will not give consistent J_z values. This limits the number of spin-3/2 baryons to 10, depicting the decuplet representation of $SU(3)$.

However, in the case of mesons, we can add to the two-pion configuration ($\pi^+\pi^0$) ($\pi^-\pi^0$) with $l_z^{\pi^0} = -1/2$ (+1/2) another neutral π^0 having $l = 1/2$, $l_z = -1/2$ (+1/2) for the particle (antiparticle) state. This will lead to the isosinglet state ($\pi^+\pi^0\pi^0$) having strangeness $(-1/2)S = l_z^{\pi^0} + l_z^{\pi^0} = -1/2 - 1/2 = -1$, implying $S = +2$. Evidently this $I = 0$, $S = +2$ state will correspond to a positively charged meson. We can get the antiparticle state ($\pi^-\pi^0\pi^0$) with opposite l_z values, which will give rise to negatively charged meson having $I = 0$, $S = -2$. Apart from these states, the system ($\pi^0\pi^0\pi^0$) with one $l_z^{\pi^0} = +1/2$ and another $l_z^{\pi^0} = -1/2$ will lead to an isosinglet neutral meson with $S = 0$. This arises due to the fact that the particle and antiparticle states in this case cannot be distinguished and so there can be a mixing of $l_z^{\pi^0} = -1/2$ (+1/2) with $l_z^{\pi^0} = +1/2$ (−1/2). However, no other π^0-meson having the constraint $l_z = -1/2$ (+1/2) for particle (antiparticle) configurations can be accommodated in this scheme, as this will not lead to consistent J_z values. This suggests that apart from the octet of mesons, we can have three pseudoscalar mesons D^+, D^0, D^- having isospin 0 and strangeness $+2$, 0, and -2, respectively. For vector mesons, the neutral configuration is the well-known ϕ^0-meson. From the equal spacing rule we find $m_D \simeq 725$ MeV and $m_\phi \simeq 1020$ MeV. The isoscalar charged meson D^+ with strangeness $+2$ was reported by Yamanouchi [21] and Yamanouchi and Kaplan [22], who also suggested the same mass ($\simeq$ 730 MeV) for these particles.

With the introduction of a spinor with $l = 1/2$, $l_z = +1/2$ in the configuration of these mesons we get exotic baryons. Indeed, the configuration $(\pi^+\pi^0\pi^0\nu_\mu)$ with $J^{\nu_\mu} = 0$, $l_z^{\pi^0} = -1/2$ will lead to the spin $-$ 1/2 baryon Z_0^{*+} having $I = 0$, $S = +1$ and for the configurations $(\pi^+\pi^0\pi^0\mu^+)$, $(\pi^+\pi^0\pi^0\nu_\mu)$, $(\pi^0\pi^0\pi^0\nu_\mu)$ with J^{μ^+} $(J^{\nu_\mu}) = 1$, $J_z^{\mu^+} = +1$, $J_z^{\nu_\mu} = 0$ we get spin $-3/2$ baryons Z_1^{*++}, Z_1^{*+}, Z_1^{*0} having $I = 1$ and $S = +1$.

We can construct configurations for $J^{PC} = 0^{++}$, 1^{++}, 1^{+-} mesons and high-spin baryons by incorporating one π^0, ρ^0, ω^0, or f^0 in the S state of the above configurations of mesons and baryons. This will not alter any internal quantum number and will change only spin and parity.

In mesonic and baryonic multiplets the members having different strangeness with $|\Delta S| = 1$ are characterized by having one extra pionic constitutent with its mass modified by the associated magnetic field energy. This will automatically lead to the equal spacing rule as suggested by the Gell-Mann–Okubo formula. One interesting consequence of this configuration scheme is that vector mesons as well as spin-3/2 baryons can be characterized by the fact that the constituents have relative angular momentum $L = 1$, whereas for pseudoscalar mesons and spin-l/2 baryons we have $L = 0$. So these will satisfy the mass relations

$$\begin{aligned} m_\rho^2 - m_\pi^2 &= m_{K*}^2 - m_K^2 \\ &= m_{N*}^2 - m_N^2 \\ &= m_{\Sigma*}^2 - m_\Sigma^2 \\ &= m_{\Xi*}^2 - m_\Xi^2 \end{aligned} \tag{63}$$

which are known to be in good agreement with experiments.

5. STATIC PROPERTIES OF BARYONS

As mentioned earlier, since in the Skyrme model the Skyrme term as well as the Wess–Zumino term appears as an effect of the anisotropic feature of the microlocal space-time and is associated with the quantization of a fermion, these may be treated as representing quantum fluctuations. We can now compute the static properties of baryons from the point of view that these are composites of baby skyrmions using the following Lagrangian for a baby skyrmion represented by pionic degrees of freedom:

$$L = \frac{1}{16} F_\pi^2 \, \partial_\mu \, U^\dagger \, \partial_\mu \, U + \frac{1}{32e^2} Tr[\partial_\mu UU^\dagger, \, \partial_\nu UU^\dagger]^2 \tag{64}$$

As baby skyrmions take only pionic degrees of freedom and $SU(3)$ symmetry is generated in a specific geometrical setup, for computation we have restricted ourselves to the $SU(2)$ case. In case of $SU(2)$, the Wess–Zumino term(19) vanishes. For a composite model of baby skyrmions, only the kinetic term in the Lagrangian will be modified depending on the number of baby skyrmions. This is because, as mentioned in Section 2, the origin of the second term is the anisotropy of the internal space. So for a composite system where the constituents (baby skyrmions) are taken to move with $\ell = 1/2$ in an anisotropic space, this second term will just represent the overall anisotropic feature of the internal space of the composite system. So the whole effect of the different number of baby skyrmions for different composite states representing various baryons will have to be incorporated in the kinetic term.

As discussed in Section 2, the topological features associated with chiral anomaly relate the second component of the $SL(2, C)$ gauge field current j_μ^2 with the axial vector current j_μ^5 through the relation [Eq. 33]

$$\partial_\mu j_\mu^5 = -2\partial_\mu j_\mu^2$$

In view of this, we note that the pion decay constant F_π which is associated with the axial vector current j_μ^5 is related to the topological current $2j_\mu^2$ through this relation, where j_μ^2 represents the Chern–Simons characteristic class. So this topological relation suggests that for one baby skyrmion, the term F_π in the Lagrangian (64) should be replaced by $F_\pi/2$. Since a nucleon is taken to be composed of two such baby skyrmions (pions) with a spinor attached to them, we should replace the term F_π^2 by $(F_\pi^2/4) \cdot 2 = F_\pi^2/2 = F_\pi'^2$, where F_π is the experimental value of the pion decay constant. Similarly, for Λ, $\Sigma(\Xi)$, which is considered to be composed of 3 (4), baby skyrmions (pions) with a spinor attached to them, the value of F_π^2 in Eq. (64) should be replaced by $F_\pi''^2$ $(F_\pi'''^2)$, where $F_\pi''^2 = \frac{3}{4}F_\pi^2$ and $F_\pi'''^2 = \frac{4}{4}F_\pi^2 = F_\pi^2$. In view of this, we can incorporate the effect of different number of baby skyrmions in various composite states depicting different baryons in the modified value of F_π^2 in Eq. (64).

5.1. Mass Spectrum of Baryons

We may now follow Adkins *et al.* [23] to compute the mass of baryons. We take as input the experimental value of the pion decay constant $F_\pi =$

186 MeV. From the Lagrangian (64) we find the soliton solution by using the Skyrme ansatz

$$U_0(x) = \exp[iF(r)\,\vec{\tau}\cdot\vec{x}] \tag{65}$$

where $F(r) = \pi$ at $r = 0$ and $F(r) \to 0$ as $r \to \infty$. If we substitute this ansatz in (64), we get the expression for the soliton mass

$$\tilde{M} = 4\pi \int_0^\infty r^2 \left\{ \frac{1}{8} F_\pi^2 \left[\left(\frac{\partial F}{\partial r} \right)^2 + \frac{2\sin^2 F}{r^2} \right] + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left[\frac{\sin^2 F}{r^2} + 2 \left(\frac{\partial F}{\partial r} \right)^2 \right] \right\} dr \tag{66}$$

Now if $U_0 = \exp(iF(r)\tilde{\tau}\cdot\tilde{x})$ is the soliton solution, then $U = AU_0 A^{-1}$, where A is an arbitrary, constant, $SU(2)$ matrix, is a finite-energy solution as well. To treat A as a collective coordinate so that it behaves as a quantum mechanical variable, we take

$$U = A(t) U_0 A^{-1}(t) \tag{67}$$

where $A(t)$ is an arbitrary time-dependent $SU(2)$ matrix. From this we get [23]

$$L = -\tilde{M} + \lambda\, Tr\,[\partial_0 A \partial_0 A^{-1}]$$

where $\tilde{M}$ is defined in (66) and $\lambda = \frac{4}{6}\pi\,(1/e^3 F_\pi)\,\Lambda$ with

$$\Lambda = \int \tilde{r}^2 \sin^2 F \left[1 + 4\left(F'^2 + \frac{\sin^2 F}{\tilde{r}^2} \right) \right] d\tilde{r} \tag{68}$$

with $\tilde{r} = eF_\pi r$. Numerically, $\Lambda = 50.9$. The $SU(2)$ matrix A can be written $A = a_0 + i\vec{a}\cdot(\tan)\vec{r}$ with $a_0^2 + \vec{a}^{\,2} = 1$. In terms of this, we can write

$$L = -\tilde{M} + 2\lambda \sum_{i=0}^{3} (\dot{a}_i)^2 \tag{69}$$

Introducing the conjugate $\pi_i = \partial L/\partial \dot{a}_i = 4\lambda \dot{a}_i$, we have the Hamiltonian

$$H = \pi_i \dot{a}_i - L = 4\lambda \dot{a}_i \dot{a}_i - L \tag{70}$$

Taking $\pi_i = -i\partial/\partial a_i$ as suggested by the canonical quantization procedure, we get

$$H = \tilde{M} + \frac{1}{8\lambda} \sum_{i=0}^{3} (-\partial^2/\partial a_i^2) \tag{71}$$

with the constraint $\Sigma_{i=0}^3 a_i^2 = 1$. The operator can be interpreted as the Laplacian Δ on the three-sphere. The wave functions are traceless symmetric

Table I. Mass of the Baryons

Baryon	Effective meson decay constant	Mass (MeV)
N	$F'_\pi = F_\pi/\sqrt{2}$	940 (input)
Λ, Σ	$F''_\pi = F_\pi\sqrt{3/2}$	1151
Ξ	$F'''_\pi = F_\pi$	1330

Table II. Mass of Spin-3/2 Baryons

Baryon	Mass (MeV)
N^*	1216
Σ^*	1386
Ξ^*	1537

polynomials in the a_i. A typical example is $(a_0 + ia_i)^l$ with $-\Delta\,(a_0 + ia_i)^l = l(l + 2)\,(a_0 + ia_i)^l$. Such a wave function has spin $1/2l$ corresponding to the angular momentum of the baby skyrmion. Now the eigenvalues of the Hamiltonian are

$$E = \tilde{M} + \frac{1}{8\lambda}\,l(l + 2) \tag{72}$$

with $l = 2j$, where j corresponds to the angular momentum of the baby skyrmion. From this, we find [23]

$$\tilde{M} = 36.5F_\pi/e$$

$$\lambda = \frac{4}{6}\,\pi\left(\frac{1}{e^3F_\pi}\right) 50.9 \tag{73}$$

Now for the composite model as considered above, we substitute the values of F_π by F'_π, F''_π, and F'''_π for N, (Λ, Σ), and Ξ particles, respectively, and we find the values in Table I for the mass of the baryons taking the nucleon mass as the input. The value of e is found to be 5.585 [24].

For spin-3/2 baryons, as discussed in the previous section, we have the mass relations

$$\begin{aligned} m_\rho^2 - m_\pi^2 &= m_{N^*}^2 - m_N^2 \\ &= m_{\Sigma^*}^2 - m_\Sigma^2 \\ &= m_{\Xi^*}^2 - m_\Xi^2 \end{aligned} \tag{74}$$

The values of masses of spin-3/2 baryons from the above relation using the value, $m_\rho = 785$ Mev and $m_\pi = 140$ Mev are given in Table II. The value of m_{Ω^-} can be found from the equal spacing rule, which gets support from the decomposition of the symmetry $SU(3) \rightarrow SU(2) \otimes U(1)$, and the value of strangeness -3 suggests that it has another baby skyrmion more than that of Ξ^*.

5.2. Magnetic Moments of Baryons

From the discussion above, we can define the anomalous baryon current as

$$B^{\mu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2} Tr[(U^{-1}\partial_{\nu}U)\,(U^{-1}\partial_{\alpha}U)\,(U^{-1}\partial_{\beta}U)] \tag{75}$$

This follows from Eq. (28), when in the pure gauge condition $F_{\mu\nu} = 0$, we can write

$$B_{\mu} = U^{-1}\partial_{\mu}U, \qquad U \in SU(2)$$

If we substitute $U = A(t)U_0A^{-1}(t)$, following Adkins *et al.* [23], we can write the angular integrals associated with the V-A current

$$\int d\Omega\; V^{a,0} = \frac{1}{3}\, i4\pi\Lambda'\; Tr[(\partial_0\, A)A^{-1}\tau_a] \tag{76}$$

$$\int d\Omega\; \vec{q}\cdot\vec{x}V^{a,i} = \frac{1}{3}\, i\pi\Lambda'\; Tr(\vec{\tau}\cdot\vec{q}\tau_i A^{-1}\tau_a A) \tag{77}$$

$$\int d\Omega\; A^{a,i} = \frac{1}{3}\, \pi D'\; Tr(\tau_i A^{-1}\tau_a A) \tag{78}$$

where

$$\Lambda' = \sin^2 F\left[F_{\pi}^2 + \frac{4}{e^2}\left(F'^2 + \frac{\sin^2 F}{r^2}\right)\right]$$

$$D' = F_{\pi}^2\left(F' + \frac{\sin 2F}{r}\right) + \frac{4}{e^2}\left(\frac{\sin 2F}{r}F'^2 + \frac{2\sin^2 F}{r^2}F' + \frac{\sin^2 F \sin 2F}{r^3}\right)$$

From (75) the baryon current and charge density can be written as

$$B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F' \tag{79}$$

$$B^i = i\frac{\epsilon^{ijk}}{2\pi^2}\frac{\sin^2 F}{r}F'\hat{x}_k\, Tr[(\partial_0 A^{-1})A\tau_j] \tag{80}$$

The baryon charge per unit r is

$$\rho_B(r) = 4\pi r^2 B^0(r) = -\frac{2}{\pi}\sin^2 F\, F'$$

and its integral $\int_0^\infty \rho_B(r)\, dr = 1$ gives the baryonic charge. The isoscalar mean square radius is given by

$$\langle r^2\rangle_{I=0} = \int_0^\infty r^2\rho_B(r)\, dr = \frac{4.47}{e^2F_\pi^2} \tag{81}$$

From (76), the isovector charge density is given by

$$\rho_{I=1}(r) = \frac{r^2 \sin^2 F\,\{F_\pi^2 + (4/e^2)[F'^2 + (\sin^2 F)/r^2]\}}{\int_0^\alpha r^2 \sin^2 F\,\{F_\pi^2 + (4/e^2)[F'^2 + (\sin^2 F)/r^2]\}\, dr} \tag{82}$$

The isoscalar and isovector magnetic moments are, respectively,

$$\vec{\mu}_{I=0} = \frac{1}{2}\int \vec{r} \times \vec{B}\, d^3x$$

$$\vec{\mu}_{I=1} = \frac{1}{2}\int \vec{r} \times \vec{V}^3\, d^3x \tag{83}$$

The isoscalar magnetic moment density is

$$\rho_M^{I=0}(r) = \frac{r^2\, F'\, \sin^2 F}{\int r^2\, F'\, \sin^2 F\, dr} \tag{84}$$

and the isoscalar magnetic mean radius is defined by

$$\langle r^2\rangle_{M,I=0} = \int_0^\infty r^2\, \rho_M^{I=0}\,(r)\, dr \tag{85}$$

The isoscalar magnetic moment is

$$(\mu_{I=0})_3 = \frac{\langle\tilde{r}^2\rangle_{I=0}}{\Lambda}\frac{e}{F_\pi}\frac{1}{4\pi} \tag{86}$$

where $\tilde{r} = eF_\pi r$. Also, for the isovector magnetic moment we have [23]

$$(\mu_{I=1})_3 = \frac{2}{9}\,\pi\,\frac{\Lambda}{F_\pi e^3} \tag{87}$$

Now, to incorporate the change in the value of F_π as suggested in Table I, we note that the current B^μ in Eq. (75) is identical with the current j_μ^2, the second component of the gauge field current when the gauge group is taken to be $SU(2)$, and we use the asymptotic pure gauge condition $B_\mu = U^{-1}\,\partial_\mu U$. However, the integral $\int j_0^2\, d^3 x$ corresponds to the monopole charge μ, which is related to the baryonic charge given by the winding number q of the mapping of the group manifold $S^3[SU(2) = S^3]$ to the field manifold S^3 by the relation $2\mu = q$ [13]. For the baryonic charge 1, we have $\mu = 1/2$. So we should normalize the expression for the mean square radius $\langle r^2\rangle$ accordingly. Now, from expression (81) we note that this should be multiplied by 2, which we can incorporate by changing the factor F_π^2 to $F_\pi^2/2$. This change should be incorporated in the isovector case also. Again as suggested in Table I, due to the compositeness of nucleon, we should change $F_{\overline{\pi}}$ to $F'_\pi = F_\pi/\sqrt{2}$. Thus the effective change in the expressions for the isoscalar and isovector magnetic moment, (86) and (87) is to replace F_π by $F_\pi/\sqrt{2}$, where F_π is the experimental value of the pion decay constant (186 MeV). The g factor is defined by writing

$$\vec{\mu} = \left(\frac{g}{4M}\right)\vec{\sigma}$$

Now incorporating the above changes in the value of F_π, we find the isoscalar g factor

$$g_{I=0} = g_p + g_n = 1.56 \tag{88}$$

and the isovector g factor

$$g_{I=1} = g_p - g_n = 8.16 \tag{89}$$

From these, we find $g_p = 4.86$ and $g_n = -3.30$ This suggests

$$\begin{aligned} \mu_p &= 2.43 \\ \mu_n &= -1.65 \end{aligned} \tag{90}$$

and the ratio $|\mu_p|/|\mu_n| = 1.47$, which is to be contrasted with the value 1.5 in the naive quark model. To compute the magnetic moments of hyperons, we note that as the configuration of Λ, Σ (Ξ) has been taken to be composed of three (four) baby skyrmions, which is to be compared with the two-baby-skyrmion model of a nucleon apart from the neutral spinorial constituent, we can write the configurations of $\Sigma^+ = (p\pi^0)$, $\Sigma^- = (n\pi^-)$, $\Lambda = (n\pi^0)$, $\Xi^- = (\Sigma^-\,\pi^0)$, and $\Xi^0 = (\Sigma^0\,\pi^0)$. The correction factor due to the change in the value of F_π as given in Table I which is to be incorporated in the expressions of the magnetic moment is .815 for Λ, Σ in relation to that of the nucleon,

$$\frac{1}{F''_\pi} = \frac{2}{\sqrt{3}} \cdot \frac{1}{F_\pi} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{2}}{F_\pi} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{F'_\pi} = .815\ 1/F'_\pi$$

and .865 for Ξ particles in relation to that of Λ Σ,

$$\frac{1}{F'''_\pi} = \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}} \frac{1}{F_\pi} = \frac{\sqrt{3}}{2} \cdot \frac{1}{F''_\pi} = .865\ \frac{1}{F''_\pi}$$

we can now compute the magnetic moments of hyperons. For this configuration scheme, we find

$$\mu_{\Sigma^+} = \mu_p \times .815 = 2.43 \times .815 = 1.98$$
$$\mu_{\Sigma^-} = \mu_n \times .815 = -1.65 \times .815 = -1.35 \tag{91}$$

Again, since we have $\Lambda = (n\pi^0)$ and the magnetic moment of the neutron is negative, we write

$$\begin{aligned}(\mu_\Lambda)_{I=0} &= -(\mu_p + \mu_n) \times .815 \\ &= -(2.43 - 1.65) \times .815 \\ &= -.63\end{aligned} \tag{92}$$

Similarly, we find

$$\mu_{\Xi^-} = \mu_{\Sigma^-} \times .865 = -\ 1.35 \times .865 = -1.17 \tag{93}$$

Noting that the configuration of Ξ^0 is given by $(\Sigma^0\pi^0)$, where for $(\Sigma^0\pi^0)$ we again write $(n\pi^0\ \pi^0)$, we find

$$\mu_{\Xi^0} = \mu_n \times .7$$

where .7 is the conversion factor in F_π in relation to the nucleon,

$$\frac{1}{F'''_\pi} = \frac{1}{F_\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{F_\pi} = .7\ \frac{1}{F'_\pi}$$

From this we have

$$\mu_{\Xi^0} = -1.65 \times 0.7 = -1.15 \tag{94}$$

In Table III we display the predicted values for comparison with the experimental values.

Table III. Magnetic Moments of Baryons

Magnetic moment	Predicted value	Experimental value
μ_p	2.43	2.79
μ_n	-1.65	-1.91
$\lvert\mu_p\rvert, \lvert\mu_n\rvert$	1.47	1.46
μ_{Σ^+}	1.98	2.42
μ_{Σ^-}	-1.35	-1.16
μ_Λ	$-.63$	$-.61$
μ_{Ξ^-}	-1.17	$-.69$
μ_{Ξ^0}	-1.15	-1.25

5.3. Electromagnetic Mass Difference

From the composite nature of baryons, we can now find the electromagnetic mass difference of an isomultiplet of baryons. Indeed, as in our scheme l_z values of baby skyrmions are associated with strangeness, for a nucleon where we have taken that this is composed of two baby skyrmions with a spinor bound to it, and noting that two baby skyrmions may lead to a strange particle like a K-meson, where strangeness is generated by the l_z value of a baby skyrmion, this strangeness value is canceled by the l_z value of the spin carrier, making it a nonstrange particle. So, assuming the simplest choice of equal probability for various configuarations, we can take the configuration of a nucleon as

$$N = \frac{1}{\sqrt{2}}\,[(\pi\pi\nu_\mu) + (K\nu_\mu)]$$

So for the proton and the neutron we can write

$$p = \frac{1}{\sqrt{3}}\,[(\pi^+\,\pi^0\,\nu_\mu) + (\pi^+\,\pi^0\,\nu_\mu) + (K^+\,\nu_\mu)]$$
$$n = \frac{1}{\sqrt{3}}\,(\pi^+\,\pi^-\,\nu_\mu) + (\pi^0\,\pi^0\,\nu_\mu) + (K^0\,\nu_\mu)]$$

Thus the mass difference is given by

$$\begin{aligned} m_p - m_n &= \tfrac{1}{3}[(m_{\pi^0} - m_{\pi^-}) + (m_{\pi^+} - m_{\pi^0}) + (m_{K^+} - m_{K^0})] \\ &= \tfrac{1}{3}\,(m_{K^+} - m_{K^0}) = \tfrac{1}{3}\,(-4)\text{ MeV} = -1.3\text{ MeV} \end{aligned} \tag{95}$$

which is in excellent agreement with experiment, with the correct sign (Table

IV). Similarly for the mass differences $\Sigma^+ - \Sigma^0$ and $\Sigma^0 - \Sigma^-$ we note that a Σ-baryon is composed of three baby skyrmions with a spin carrier, so we write

$$\sum = (\pi\pi\nu_\mu\pi)$$

where we note that the configuration $(\pi\pi)$ may give rise to a kaon and $(\pi\pi\nu_\mu)$ gives rise to a nucleon. Thus we can write explicitly

$$\sum^+ = \frac{1}{\sqrt{2}}\,[(K^+\,\nu_\mu\pi^0) + (p\pi^0)]$$

$$\sum^0 = \frac{1}{\sqrt{2}}\,[(K^0\nu_\mu\pi^0) + (n\pi^0)]$$

So

$$m_{\Sigma^+} - m_{\Sigma^0} = \frac{1}{2}\,[(m_{K^+} - m_{K^0}) + (m_p - m_n)]$$
$$= \frac{1}{2}\,[-4 - 1.3]\text{ MeV} = \frac{-5.3}{2}\text{ MeV} = -2.65\text{ MeV} \qquad (96)$$

which is to be compared with the experimental value, -3 MeV.

For $m_{\Sigma^0} - m_{\Sigma^-}$, we note that as we have no negatively charged nucleon, we can write the configuration of Σ^- as $(n\pi^-)$. Again the $(\pi\pi)$ system here also will not represent a negatively charged kaon K^-, as this along with the spin carrier would have created a negatively charged nucleon. Hence we write

$$\sum^0 = \frac{1}{\sqrt{2}}\,[(K^0\;\nu_\mu\;\pi^0) + (n\pi^0)]$$

$$\sum^- = \frac{1}{\sqrt{2}}\,[(K^0\;\nu_\mu\;\pi^-) + (n\pi^-)]$$

so that

Table IV. Electromagnetic Mass Difference of Baryons

Mass difference	Predicted value (MeV)	Experimental value (MeV)
$m_p - m_n$	-1.3	-1.3
$m_{\Sigma^+} - m_{\Sigma^0}$	-2.65	-3
$m_{\Sigma^0} - m_{\Sigma^-}$	-4.6	-4.8
$m_{\Xi^0} - m_{\Xi^-}$	-6.4	-6.4

$$m_{\Sigma^0} - m_{\Sigma^-} = \frac{1}{2}\left[(m_{\pi^0} - m_{\pi^-}) + (m_{\pi^0} - m_{\pi^-})\right] = -4.6 \text{ MeV} \quad (97)$$

which is in excellent agreement with the experimental value, -4.8 MeV.

A similar analysis suggests that Ξ^0, Ξ^- may be depicted as a composite of $\Sigma\pi$ with proper charge distribution. Indeed, Ξ^0 and Ξ^- can be written as a combination of states

$$\Xi^0 = \sum^+ \pi^- + \sum^- \pi^+ + \sum^0 \pi^0$$
$$\Xi^- = \sum^- \pi^0 + \sum^0 \pi^-$$

From this, we get

$$\begin{aligned} m_{\Xi^u} - m_{\Xi^-} &= (m_{\Sigma^+} - m_{\Sigma^-}) + (m_{\pi^-} - m_{\pi^0}) + (m_{\Sigma^+} - m_{\Sigma^0}) \\ &\quad + (m_{\pi^+} - m_{\pi^0}) + (m_{\Sigma^-} - m_{\Sigma^0}) + (m_{\Sigma^0} - m_{\Sigma^-}) \\ &\quad + (m_{\pi^0} - m_{\pi^-}) \\ &= (m_{\Sigma^+} - m_{\Sigma^-}) + (m_{\pi^-} - m_{\pi^0}) + (m_{\Sigma^+} - m_{\Sigma^0}) \end{aligned}$$

Putting in the experimental values of the mass difference, we get

$$m_{\Xi^0} - m_{\Xi^-} = (-7.9 + 4.6 - 3.1) \text{ MeV} = -6.4 \text{ MeV} \quad (98)$$

This value is also in excellent agreement with the experimental value, -6.4 ± 0.6 MeV.

In the configuration of Ξ^0 (Ξ^-) we have taken only the charge mixing effect taking into account the hybrid state of various charge combinations. The values of $m_{\Sigma^+} - m_{\Sigma^0}$, $m_{\Sigma^+} - m_{\Sigma^-}$, and $m_{\Sigma^0} - m_{\Sigma^-}$ have been computed from various configuration mixings as shown above and the probable combination of various charge states just represents a *resonating effect* depicting a hybrid configuration.

6. SOME ASPECTS OF HADRONIC INTERACTIONS

According to this model of hadrons, right-handed and left-handed systems appear in a symmetric way and this symmetry is obtained from conformal and space-time reflection; when we split the conformal spinors into a doublet of Cartan semispinors, the above reflection symmetries can only be maintained in strong interactions which involve only hadrons by the preservation of handedness in the left- and right-handed systems representing particles and antiparticles. This can be ensured through the rearrangement of the constituents in such a way that the specific handedness for particle and antiparticle systems is not altered. This implies that strong interactions involve composite systems of elementary spinors in the configuration of a hadron. The symmetry principle is built into the very dynamics of strong interactions. In an earlier paper [25] we showed that the rearrangement of constituents with specific handedness for particle and antiparticle systems can remove all the inconsistencies which crop up in the naive field-theoretic formalism and satisfy all the canons of *S*-matrix theory. Also this satisfies the properties of duality. Indeed, it has been shown that according to this formalism any strong interaction without exchange of hypercharge can be explained in terms of $\pi\pi$ interaction, where the interacting pions are composite states of elementary fermions in the structure of a hadron along with the rearrangement of the constituents. This rearrangement of the constituents gives rise to a Regge type of amplitude for the process. Again, for a hypercharge-changing process we suggested that this may occur when a constituent pion contributing to the internal quantum numbers is knocked out and taken by a hadron in such a way that the handedness is not altered. This will ensure the conservation of hypercharge.

In case an elementary spinorial system take part in an interaction in Minkowski space, it is expected that the right–left symmetry as well as the full internal symmetry which appear here as a manifestation of the composite character of conformal spinors will be destroyed. This is the case for weak interaction, where parity, charge conjugation, and isospin symmetry are not maintained. Indeed, in this scheme all $\Delta S = 0$ semileptonic decay processes follow from the decay of the muon in the structure of the hadron and this explains the universality of the weak decay coupling constant. Indeed this is the main motivation behind taking muonic leptons as the constituents of hadrons. For example, β-decay can be formulated through the decay of μ^- in the configuration

$$n = (\mu^- \ \mu^+ \ \pi^0 \ \nu_\mu) \to (\nu_\mu \ \mu^+ \ \pi^0 \ \nu_\mu) + e^- + \bar{\nu}_e = p + e^- + \bar{\nu}_e$$

Again it is suggested that semileptonic decays with $|\Delta S| = 1$ occur as a result of the decay of the muon as well as the transition $\pi^0 \to$ vacuum, where the pionic constituent in the structure of the hadron contributes to the isospin and strangeness of the hadron concerned. For example, the $\Lambda \to pe^- \bar{\nu}_e$ can be interpreted as follows:

$$\begin{aligned}\Lambda &= (\mu^- \ \mu^+ \ \pi^0 \ \nu_\mu \ \pi^0) \to (\nu_\mu \ \mu^+ \ \pi^0 \ \nu_\mu) + e^- + \bar{\nu}_e + (\pi^0 \to \text{vacuum}) \\ &= p + e^- + \bar{\nu}_e\end{aligned}$$

This explains nicely the selection rules $|\Delta I| = 1/2$, $|\Delta S| = 1$, and $\Delta S/\Delta Q \neq -1$ from the very dynamics of weak processes. Also, for nonleptonic decay processes the $|\Delta I| = 1/2$ and $|\Delta S| = 1$ rules follow from the dynamics of such processes. Indeed, such processes occur when the transition $\pi^0 \to$

vacuum occurs and the residual system decays into relevant hadrons, which can be treated nicely through the pole approximation method [26]. This mechanism incorporates the above selection rules.

An interesting implication of this scheme is that since the internal symmetry algebra is generated from reflection group, parity conservation in strong interaction is found to be a consequence of isospin conservation. Again parity as well as charge conjugation violation in the weak interaction are consequences of isospin violation in such processes. In case of electromagnetic interactions, isospin violation is not accompanied by parity violation due to the fact that this reflection group demands a disconnected gauge group for such an interaction when the current is written in chiral form [13].

We have considered eight-component conformal spinors as the basic ingredient of internal symmetry, so that when these split into two four-component spinors in Minkowski space, these take part as the constituents of hadrons in such a way that the members of the doublet take part in particle and antiparticle configurations. Haag *et al.* [27] discussed the interplay between the conformal symmetry, internal symmetry, and supersymmetry. Daniel and Ktorides [28] adopted $R^{4,2}$ as the underlying space of supersymmetry and considered the supersymmetry algebra as the algebra of inhomogenous rotation in the spinor space associated with $R^{4,2}$ plus an additional number of generators which can be readily interpreted as the elements of $U(n)$ algebra. In their attempt to construct the algebra, they derived the following relation for the anticommutators of two conformal spinors:

$$\{Q_\alpha, Q_\beta\} = \eta_{ABC}(\gamma_{ABC}\, J)_{\alpha\beta} + \eta_{AB}\,(\gamma_{AB}\, J)_{\alpha\beta} + \eta(\beta_7 J)_{\alpha\beta}$$

where $\gamma_{ABC} = \beta_A\beta_B\beta_C$, β_A, *s* are the 8×8 matrices in C_6 representing the unit vectors in $R^{4,2}$, $\beta_7 = \beta_0\beta_1\beta_2\beta_3\beta_5\beta_6$, and $\gamma_{AB} = \frac{1}{2}(\beta_A\beta_B - \beta_B\beta_A)$. The parameters η_{ABC}, η_{AB}, and η are mapped onto the group generators. η_{AB} effectively corresponds to J_{AB}, the 15 generators of the conformal group $SU(2, 2)$. The parameter η is mapped onto a pseudoscalar generator which is identified as the the γ_5 transformation and is responsible for the generation for the internal symmetry. η_{ABC} corresponds to reflection (rotation + reflection). Daniel and Ktorides ignored this. However, as it has been shown that the reflection may lead to U_2 algebra, we note that the anticommutator of conformal spinors leads to the algebra

$$S = g \times C$$

where g is the internal symmetry algebra given by

$$g = \{(SU(2) \times U(1))_L \oplus (SU(2) \times U(1)_R \oplus U(1)\}$$

where $U(1)$ corresponds to strong reflection and C is the conformal algebra given by $SU(2, 2)$. Thus the maximal internal symmetry we may observe in one interaction is given by g. Since conformal reflection gives rise to isospin algebra $SU(2)$ when for particle and antiparticle systems we have two independent $SU(2)$ algebras and hypercharge is given by $U(1)$ algebra such that it has opposite values for these systems, strong interaction symmetry $SU(3)$ is manifested when the mass splitting of hadrons is given by the symmetry breaking $SU(3) \rightarrow SU(2) \times U(1)$, and for the particle and antiparticle world we can write it as $(SU(2) \times U(1))_L$ and $(SU(2) \times U(1))_R$, respectively. Besides this we can have a parity-violating interaction $SU(2)_L \times U(1)$ [or $SU(2)_R \times U(1)$], which is the symmetry group of the electroweak interaction. Indeed it has been shown in a recent paper that the topological properties of a fermion help us to realize the $SU(2)_L \times U(1)$ group structure for electroweak unification, and weak interaction gauge bosons attain their masses, which are of topological origin [29]. When fermions are written in chiral form, the electromagnetic interaction is characterized by the disconnected gauge group $U(1)_L \times U(1)_R$ instead of the group $U(1)$ [30, 13]. When the full symmetry group g is taken into account, this is found to be related to gravitation given by the Einstein–Cartan action when the $U(1)$ corresponding to strong reflection gives rise to torsion, which appears as the contribution of quantum gravity [31]. As the metric part of gravitation does not distinguish between particles and antiparticles, the internal symmetry algebra $\{(SU(2) \times U(1))_L \oplus (SU(2) \times U(1))_R\}$ is not disturbed by it. These are the only four possibilities we have from the group structure g which respect CP invariance. This explains why we have only four types of interactions in nature. Moreover, since the algebra $g \times C$ gives rise to supersymmetry algebra S, we can have supersymmetric phase only in the massless state [32] and the generation of mass is associated with the generation of internal helicity, which distinguishes bosons and fermions.

According to this model of hadrons, $SU(3)$ symmetry is found to be the maximal internal symmetry of hadrons and no flavors like charm, bottom, and top can be accommodated in this picture. The interpretation of ψ and γ particles in terms of $c\bar{c}$ and $b\bar{b}$ bound states is not beyond ambiguity, as it cannot explain consistently all the decay modes with their proper widths and mass differences, as has been emphasized by many authors. The interpretation of D-mesons as 'charmed' mesons is also in trouble with respect to experimentally observed relations like $\Gamma(D^0) \gg \Gamma(D^+)$. Order-of-magnitude disagreements have been found between old predictions and new measurements of ψ and γ production at several collider facilities [33]. So we should search for their origin in other heavy fermion models. Indeed, in a recent note [34] we showed that ψ, γ as well as *D*-mesons can indeed be taken as bound states of heavy fermions.

Another significant feature of this formalism is that baryon number-nonconserving processes like proton decay are forbidden by the requirement of Lorentz invariance. In this scheme mesonic configurations are distinguished from baryonic configurations by the fact that in the former case two constituents appear with handedness opposite to each other, and as such there is no intrinsic handedness or orientation bearing signature, but for baryons, constituents appear with a specific handedness such that this particular orientation is related to the baryon number. Since the configuration scheme suggests *CP* and *CPT* symmetry as particles and antiparticles appear on equal footing, Lorentz invariance in the external space is manifested here through *CPT* invariance. Now this *CPT* symmetry suggests that the orientation of the baryonic configurations must be preserved. In fact, if this orientation is destroyed in any process, particle–antiparticle symmetry will be destroyed. Thus baryon number conservation is found to be a consequence of *CPT* invariance and hence of Lorentz invariance. So proton decay is forbidden in this scheme by the requirement of Lorentz invariance. However, at extremely high temperature a proton can disintegrate into its constituents through a Lorentz-noninvariant interaction.

7. DISCUSSION

We have proposed a model of hadrons on the basis of the idea that the internal space is anisotropic in nature when the constituents appear as baby skyrmions, where the associated magnetic field gives rise to strong statistical attraction and the internal symmetry is generated from the reflection group. In this scheme strong interactions involve composite systems of elementary spinors in the configuration of a hadron when an elementary constituent can take part only in parity-violating weak interaction. This prompted us to take leptons as the constituents of hadrons. However, we have taken muonic leptons as the constituents from the consideration that all semileptonic decays of hadrons can be interpreted in terms of the muon decay, suggesting universality. Due to the very tiny mass of electron, it may not be possible for the $(e\nu_e)$ system to form a bound state through statistical interaction as discussed here. Again, despite e–μ–τ universality, since the τ-lepton decays into hadrons also, the configuration of τ is likely to be different from e and μ, and very probably τ itself represents a bound state. In view of this, muons seem to be the only candidate for the constituents of a hadron. This also explains the utility of the existence of muons in nature even though they behave as electrons in all interactions. In fact, a crucial question in particle physics is why muons exist at all when they behave as electrons in all aspects, and this gets a very good answer from our point of view because unless muons existed, the universe would be devoid of hadrons.

Now we summarize some of the crucial predictions of the model.

1. The observation of strangeness ± 2 and isoscalar vector particles $\phi^{\pm}$ having mass $\simeq$ 1020 MeV. The corresponding pseudoscalar particles $D^{\pm}$ with 725 MeV have already been reported by several authors [21, 22]. Particles with similar quantum numbers should also occur in other multiplets of mesons.

2. The mean square charge per constituent for p, $\Sigma^{\pm}$, and Ξ^{-} will be 1/5, 1/7, and 1/9 respectively, as is evident from the configurations. For protons it is in agreement with the experimental value, 0.18.

3. Slight breakdown of μ–e universality in ep and μp scattering as well as vector and pseudoscalar mason decay [35–37]. This will happen due to the fact that since muons (μ^{+}, ν_{μ}, μ^{-}) have been taken to be the fundamental constituents of hadrons, the basic interaction in high-energy ep and μp scattering will be effectively $e\mu$ and $\mu\mu$ scattering, and hence a slight departure from universality is expected.

4. The possible existence of a μ^{-} p resonance [38].

5. From the configuration of a proton $p = (\pi^{+}\ \pi^{0}\ \nu_{\mu})$ with $J^{\nu_{\mu}} = 0$, the spin of the proton mainly arises from the orbital momentum of the constituents. This prediction is in agreement with the recent results obtained from European Muon Collaboration [39], as pointed out by Ellis and Karliner [40]. As the configuration $(\pi^{+}\ \pi^{0})$ gives rise to a kaon, the strange degrees of freedom in a nucleon are nonvanishing in conformity with recent experimental results [41].

6. At very high energy heavy-ion collisions, a large amount of muons and neutrinos will be emitted due to the randomization of the direction vectors. This may be responsible for the large amount of neutrinos observed from Supernova ‘87.

7. Proton decay is forbidden by Lorentz invariance and at very high energy, a proton will be dissociated into $\mu^{+} + 4\nu_{\mu}$ through a Lorentz-noninvariant interaction.

8. *CP* violation of very small magnitude may occur in neutral baryons because the present scheme suggests the existence of a component having K^{0} $(\overline{K}^{0})$ in the configuration. For example, in case of a neutron $(\pi^{0}\ \pi^{0}\ \nu_{\mu})$, clustering like $(K^{0}\ \nu_{\mu})$ is possible when the $(\pi^{0}\ \pi^{0})$ system may appear as a K^{0} state. The nonzero value of the dipole electric moment of the neutron can thus be explained [42].

9. Finally, the most crucial prediction of the model is that at high density, a system of nucleons will exhibit superfluidity. Indeed, the aniso-

tropic feature of the internal space is the basic ingredient of superfluidity, as has been much discussed in literature. This may have some significant effect in neutron stars.

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REFERENCES

1. T. H. R. Skyrme (1961). *Proc. Soc. A* **260**, 127; (1962). *Nucl. Phys.* **31**, 556.
2. D. Finkelstein (1966). *J. Math. Phys.* **7**, 1218. D. Finkelstein and J. Rubinstein (1968). *J. Math. Phys.* **9**, 1762.
3. P. Bandyopadhyay and K. Hajra1 (1987). *J. Math. Phys.* **28**, 71.
4. E. Witten (1983). *Nucl. Phys. B* **223**, 422, 433.
5. K. Hajra and P. Bandyopadhyay (1991). *Phys. Lett. A* **155**, 7.
6. P. Bandyopadhyay (1989). *Int. J. Mod. Phys. A* **4**, 4449.
7. T. Shirafuji (1983). *Prog. Theor. Phys.* **70**, 18.
8. M. Fierz (1944). *Helv. Phys. Acta* **17**, 27.
9. C. A. Hurst (1968). *Ann. Phys.* **50**, 51.
10. F. Wilczek (1982). *Phys. Rev. Lett.* **48**, 1146.
11. A. Bandyopadhyay, P. Chatterjee, and P. Bandyopadhyay (1986). *Gen. Rel. Grav.* **18**, 1193.
12. M. Carmeli and S. Malin (1977). *Ann. Phys.* **103**, 206.
13. A. Roy and P. Bandyopadhyay (1989). *J. Math. Phys.* **30**, 2366.
14. A. O. Barut and A. Bohm (1970). *J. Math. Phys.* **11**, 2938.
15. E. Cartan, *The Theory of Spinors* (Paris, 1966).
16. R. Penrose (1968). *Int. J. Theor. Phys.* **1**, 61.
17. S. Sternberg (1987). *Common. Math. Phys.* **109**, 649.
18. P. Budinich (1979). *Nuovo Cimento A* **53**, 31.
19. J. P. Elliott (1958). *Proc. Soc. A* **245**, 128, 562.
20. P. Bandyopadhyay (1989). *Int. J. Mod. Phys. A* **4**, 4069.
21. Y. Yamanouchi (1959). *Phys. Rev. Lett.* **3**, 480.
22. T. Yamanouchi and M. F. Kaplan (1959). *Phys. Rev. Lett.* **3**, 203.
23. G. S. Adkins, C. R. Nappi, and E. Witten (1983). *Nucl. Phys. B* **228**, 552.
24. S. Mukherjee and P. Bandyopadhyay, Geometrical aspects of skyrmions, composite model and the static properties of baryons, Preprint.
25. P. Bandyopadhyay and S. S. De (1975). *Nuovo Cimento A* **27**, 294.
26. A. Maity and P. Mahato (1991). *Nuovo Cimento A* **140**, 269; A. Maity (1994). *Int. J. Mod. Phys. A* **9**, 3353.
27. R. Haag, J. Lopuszanski, and M. Sohnius (1975). *Nucl. Phys. B* **88**, 257.
28. M. Daniel and C. N. Ktorides (1976). *Nucl. Phys. B* **115**, 319.
29. P Bandyopadhyay, Topological aspects of chiral anomaly, origin of mass and electroweak theory, *Int. J. Mod. Phys. A* (in press).
30. J. Kiskis (1978). *Phys. Rev. D* **17**, 3196.
31. P. Bandyopadhyay, In *Modern Problems of Theoretical Physics*, P. I. Pronin and Yu. N. Obukhov, eds. (World Scientific, Singapore, 1991).
32. P. Bandyopadhyay and P. Ghosh (1989). *Int. J. Mod. Phys, A* **4**, 3791.
33. F. Abe *et al.* (1995). *Phys. Rev. Lett.* **75**, 4358.

34. P. Bandyopadhyay, Similarities between ψ and γ particles and some characteristic features of their decay modes, Preprint.
35. L. Cassilleriet *et al.* (1969). *Phys. Rev. Lett.* **23**, 153.
36. T. J. Braumstein *et al.* (1972). *Phys. Rev. D* **6**, 106.
37. D. Bryman and C. Piccioto (1975). *Phys. Rev. D* **11**, 1337.
38. I. Budagov *et al.* (1969). *Lett. Nuovo Cimento* **2**, 689.
39. European Muon Collaboration, J. Ashman *et al.* (1988). *Phys. Lett. B* **206**, 364.
40. J. Ellis and M. Karliner (1988). *Phys. Lett. B* **213**, 73.
41. H. Wiegel (1996). *Int. J. Mod. Phys. A* **11**, 2419.
42. S. Mukherjee and P. Bandyoapdhyay, Baby skyrmions, composite model and the problem of *CP* violation, Preprint.